THE GEOMETRY OF REGULAR TREES WITH THE FABER-KRAHN PROPERTY

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Abstract. In this paper we prove a Faber-Krahn-type inequality for regular trees with boundary and give a complete characterization of extremal trees. The main tools are rearrangements and perturbation of regular trees.

1. Introduction

In the last years some results for the Laplacian on manifolds have been shown to hold also for the graph Laplacian, e.g. Courant's nodal domain theorem ([2, 4]) or Cheeger's inequality ([3]). In [4] Friedman described the idea of a "graph with boundary" (see below). With this concept he was able to formulate Dirichlet and Neumann eigenvalue problems. He also conjectured another "classical" result for manifolds, the Faber-Krahn theorem, for regular bounded trees with boundary. The Faber-Krahn theorem states that among all bounded domains $D \subset \mathbb{R}^n$ with fixed volume, a ball has lowest first Dirichlet eigenvalue ([1]). Amazingly Friedman's conjecture is false, i.e. in general these trees are not "balls". First attempts to characterize extremal trees are done by the author ([5]) and with somewhat more sophisticated methods by Pruss ([6]).

In this paper we complete this characterization and extend a former result of the author.

2. Statement of the Result

Let $G(V, E)$ be an undirected graph with weights $1/c_e$ for each edge $e \in E$. The *geometric realization* of G is the metric space $\mathcal{G}$ consisting of V and arcs of length c_e glued between u and v for every edge $e = (u, v) \in E$. The *volume* $\mu(G)$ is the Lebesgue measure of $\mathcal{G}$, i.e. $\mu(G) = \sum_{e \in E} c_e$. The *Laplacian* of G is the matrix

$$\Delta = \Delta(G) = D(G) - A(G) \tag{1}$$

where $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix whose entries are the sums of the weights of the edges at the vertices of G, i.e. $D_{v,v} = \sum_{e=(v,u)\in E} \frac{1}{c_e}$. The associated Rayleigh quotient of this operator on real-valued functions f on V is the fraction

$$\mathcal{R}_G(f) = \frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} = \frac{\sum_{(u,v)\in E} \frac{1}{c_e}(f(u) - f(v))^2}{\sum_{v\in V}(f(v))^2}. \tag{2}$$

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A *graph with boundary* is a graph $G(V_0 \cup \partial V, E_0 \cup \partial E)$ with *interior vertices* V_0, *boundary vertices* ∂V and edge set $E_0 \cup \partial E$. Each edge $e \in E_0$ (*interior edge*) joins two interior vertices, each edge $e \in \partial E$ (*boundary edge*) connects an interior vertex with a boundary vertex. A *Dirichlet eigenvalue problem* can be introduced by restricting the eigenfunctions of the graph Laplacian to such f with $f(u_0) = 0$ for all boundary vertices $u_0 \in \partial V$. Equivalently we can define a Laplace operator that acts on the interior vertices of G only, i.e. on V_0:

$$\Delta_0 = D_0 - A_0 \tag{3}$$

where A_0 is the adjacency matrix restricted to V_0 and D_0 is the diagonal matrix whose entry corresponding to $v \in V_0$ is (notice $E = E_0 \cup \partial E$)

$$(D_0)_{v,v} = \sum_{e=(v,u)\in E} \frac{1}{c_e}. \tag{4}$$

Since there is no risk of confusion, we denote the Laplacian on a graph with boundary G simply by $\Delta = \Delta(G)$. For details and motivation of this definition and for basic properties of this operator see [4, 5, 6]. We denote the lowest Dirichlet eigenvalue of G by $\nu(G)$.

We are interested in regular trees with boundary. We get such a graph when we take the geometric realization of an infinite d-regular tree and cut out a bounded region. More accurately:

Definition 1. A *d-regular tree with boundary* is a tree where all interior edges have length 1 (i.e. weight 1), all boundary edges length ≤ 1, and where all interior vertices have degree d and all boundary vertices degree 1. The set of interior vertices is not empty, i.e. $|V_0| \geq 1$.

Definition 2. A *ball* $B_d(c, r)$ is a d-regular tree with boundary with a center $c \in \mathcal{G}$, not necessarily a vertex, and a radius $r > 0$, such that $\operatorname{dist}(c, v) \leq r$ for all points $v \in B_d(c, r)$, where equality holds if and only if $v \in \partial V$. $\operatorname{dist}(u, v)$ denotes the geodesic distance between $u, v \in \mathcal{G}$. $B_d(c, 0) = \{c\}$ consists of a single point.

Definition 3. We say a d-regular tree with boundary $G(V_0 \cup \partial V, E_0 \cup \partial E)$ fulfills the *Faber-Krahn-property*, if $\nu(G) \leq \nu(G')$ for every d-regular tree with boundary G' with $\mu(G') = \mu(G)$.

Unlike to the classical Faber-Krahn theorem, balls centered at a vertex does not minimize the lowest Dirichlet eigenvalue, except when all boundary edges have length 1 (see [5]).

But every tree with the Faber-Krahn-property is similar to a ball. It looks a little bit like a "peeled onion" (see figure 2). To define such a tree we need the notation of a branch.

Definition 4. Let m be the root of the tree and let $h(v) = \operatorname{dist}(m, v)$ denote the *height* of the vertex $v \in G$. Let (w, v) be an edge with $h(w) < h(v)$. The *branch* $\operatorname{Br}(w, v)$ at vertex w is the maximal subgraph induced by w, v and all descendants $u \in V$ of v (i.e. the geodesic path $(w, \ldots, u)$ contains v, see figure 1). The *length* $\ell(\operatorname{Br}(w, v))$ is the maximal distance $\operatorname{dist}(w, u_0)$, $u_0 \in \partial V$, in $\operatorname{Br}(w, v)$. The branch is called *balanced* if $h(u_0)$ is the same for all boundary vertices $u_0 \in \partial V \cap \operatorname{Br}(w, v)$.

Notice that a boundary edge is a (balanced) branch of length ≤ 1.

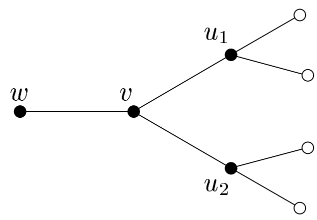

FIGURE 1. A balanced branch $\mathrm{Br}(w, v)$ of length 2.7.

Definition 5. We say a d-regular tree with boundary $G(V_0 \cup \partial V, E_0 \cup \partial E)$ is *onion shaped* if there exists a root $m \in V_0$ of the tree such that the following holds (see figure 2):

(O1) G is connected.
(O2) $B_d(m, r) \subseteq G \subseteq B_d(m, r+1)$ for an $r \in \mathbb{N}_0$ (if $|V_0| = 1$ then $r = 0$). Thus $|h(v_0) - h(u_0)| \leq 1$ for all boundary vertices $u_0, v_0 \in \partial V$.
(O3) All boundary edges have length 1 or length c, where $c \in (0, 1)$ is the same for all boundary edges of length < 1.
(O4) If two branches $\mathrm{Br}(w_1, v_1)$ and $\mathrm{Br}(w_2, v_2)$, for $h(w_1) \geq h(w_2)$, are not balanced, then $\mathrm{Br}(w_1, v_1) \subseteq \mathrm{Br}(w_2, v_2)$.

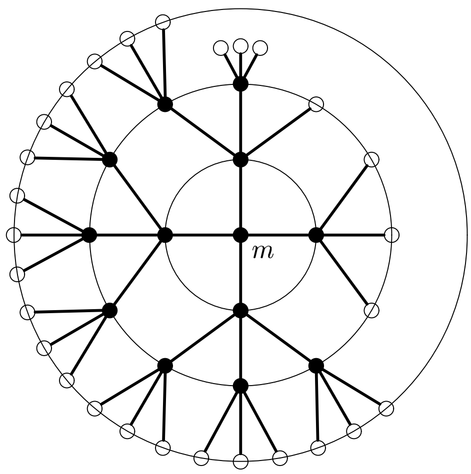

FIGURE 2. Onion shaped 4-regular tree. ($\bullet \ldots$ interior vertices, $\circ \ldots$ boundary vertices, $m \ldots$ root)

Theorem (Faber-Krahn). *A d-regular tree with boundary G, $d \geq 3$, has the Faber-Krahn property if and only if G is onion shaped and one of the following holds:*

(F0) There is only one interior vertex, i.e. $|V_0| = 1$.
(F1) All branches of length $\ell \in (1, 2]$ are balanced (this just follows from (O3)), and there is at most one balanced branch of length $\ell \in (1, 2)$, and
$d \geq 5$, or
$d = 4$ and $G \subseteq B_4(z, 4.5)$, or
$d = 3$ and $G \subseteq B_3(z, 2.5)$.

Here and in the following conditions z is the midpoint of some line in $\mathcal{G}$.

(F2) All branches of length $\ell \in (2,3]$ are balanced, and there is at most one balanced branch of length $\ell \in (2,3)$, and
$d = 4$ and $B_4(z, 4.5) \subseteq G$, or
$d = 3$ and $B_3(z, 2.5) \subseteq G \subseteq B_3(z, 9.5)$.

(F3) All branches of length $\ell \in (3,4]$ are balanced, and there is at most one balanced branch of length $\ell \in (3,4)$, and
$d = 3$ and $B_3(z, 9.5) \subseteq G$.

For a given volume μ, G is uniquely defined up to homomorphism.

Figure 3 shows the ball $B_3(z, 2.5)$. Figure 4 shows some regular trees of degree 3 of increasing volume with the Faber-Krahn property.

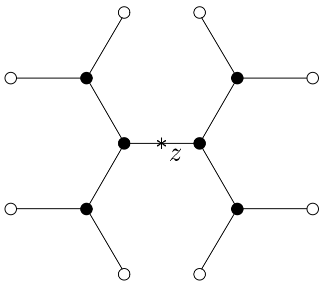

FIGURE 3. $B_3(z, 2.5)$

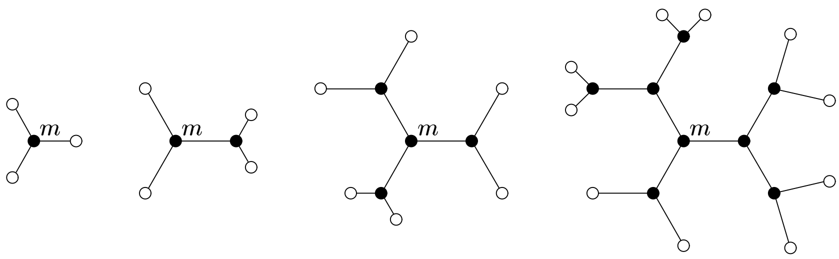

FIGURE 4. Extremal trees of degree 3. ($\bullet \ldots$ interior vertices, $\circ \ldots$ boundary vertices, $m \ldots$ root)

3. MAIN TOOLS

First we need some basic properties of $\nu(G)$.

Proposition 1 (see [4]). *Let $G(V_0 \cup \partial V, E_0 \cup \partial E)$ be a connected graph with boundary.*

(1) $\Delta(G)$ is a positive operator, i.e. $\nu(G) > 0$.
(2) An eigenfunction f to the eigenvalue $\nu(G)$ is either positive or negative on all interior vertices of G.
(3) $\nu(G)$ is continuous as a function of G in the metric $\rho(G, G') = \mu(G - G') + \mu(G' - G)$.

(4) $\nu(G)$ *is monotone in* G*, i.e. if* $G \subset G'$ *then* $\nu(G) > \nu(G')$.
(5) $\nu(G)$ *is a simple eigenvalue.*

We also need a bound on the lowest Dirichlet eigenvalue $\nu(G)$.

Proposition 2 (see [4])**.** *For a* d*-regular tree* G *with boundary we have*

$$\nu(G) > d - 2\sqrt{d-1}. \tag{5}$$

Next we extend an idea introduced in [6, definition 6.1] to regular trees with boundaries.

Definition 6. We say that a well-ordering $\prec$ on $G(V_0 \cup \partial V, E_0 \cup \partial E)$ is *spiral-like* providing the following conditions hold for all vertices $v, w, v_1, v_2, w_1, w_2 \in V_0$ and $u_1, u_2 \in \partial V$:

(S1) If $h(v) < h(w)$ then $v \prec w$.
(S2) If $v_1 \prec v_2$ and w_i is a child of v_i (i.e. $(v_i, w_i) \in E$ and $h(w_i) = h(v_i) + 1$), for $i = 1, 2$, then $w_1 \prec w_2$.
(S3) If (v_1, u_1) and (v_2, u_2) are boundary edges of lengths c_1 and c_2, respectively, with $h(v_1) = h(v_2)$ and $c_1 > c_2$, then $u_1 \prec u_2$.

Notice that in (S3) the ordering for some boundary vertices is reverse to the lengths of the incident boundary edges.

In the following let $G = G(V_0 \cup \partial V, E_0 \cup \partial E)$ be a d-regular tree with the Faber-Krahn property and f a nonnegative eigenfunction to the first Dirichlet eigenvalue $\nu(G)$. m denotes a maximum of f, i.e. $f(m) \geq f(v)$ for all $v \in V$. We always choose m for the root of our tree.

The following lemma describes the geometry of eigenfunctions to the first Dirichlet eigenvalue. It summarizes results in [5].

Lemma 3 (geometry of eigenfunctions)**.** *Let* G *be a* d*-regular tree with boundary with the Faber-Krahn property. Then*

(M1) G *is connected;*
(M2) $|h(v_0) - h(u_0)| \leq 1$ *for all boundary vertices* $u_0, v_0 \in \partial V$*;*
(M3) There exists a spiral-like well-ordering $\prec$ *such that* $u \prec v \Rightarrow f(u) \geq f(v)$*, for all vertices* $u, v \in V$.
(M4) The normal derivative of f *at all boundary edges of length* $c_e < 1$ *is the same.*

(M2) states that G is similar to a ball. Notice that it does not necessarily induce (O2). The consequence of (M3) is that f is non-increasing on every geodesic path from m to a boundary vertex and that f (nearly) has radial symmetry.

Remark. For the normal derivative we always use the orientation towards the root of the tree. Thus for a nonnegative eigenfunction it is always positive.

Corollary 4. *Let* $\mathrm{Br}(v_1, w_1)$ *and* $\mathrm{Br}(v_2, w_2)$ *be two balanced branches of lengths* ℓ_1 *and* ℓ_2*, respectively. If* $\ell_1 < \ell_2$*, then* $f(v_1) \leq f(v_2)$.

Proof. Notice that $h(v)$ is an integer for every interior vertex v. If $h(v_1) + \ell_1 \geq h(v_2) + \ell_2$, then $h(v_1) \geq h(v_2) + 1 > h(v_2)$ and by (S1) and (M3), $v_2 \prec v_1$ and $f(v_2) \geq f(v_1)$. If $h(v_1) + \ell_1 < h(v_2) + \ell_2$, then by (M2), $h(v_1) = h(v_2)$. Because of (M3) and (S3), $u_2 \prec u_1$ for boundary vertices $u_1 \in \mathrm{Br}(v_1, w_1)$ and $u_2 \in \mathrm{Br}(v_2, w_2)$. Consequently by (S2), $v_2 \prec v_1$ and $f(v_2) \geq f(v_1)$. $\square$

The main techniques for proving lemma 3 are *rearranging* and *perturbation* of edges.

Let $(v_1, u_1), (v_2, u_2) \in E$ be edges of lengths c_1 and c_2, respectively, so that u_2 is in the geodesic path from v_1 to v_2, but u_1 is not. Assume $v_1 \neq u_2$. Since G is a tree, $(v_1, v_2), (u_1, u_2) \notin E$. Thus we can replace edge (v_1, u_1) by edge (v_1, v_2) with length c_2 and edge (v_2, u_2) by edge (u_1, u_2) with length c_1 (see figure 5). Denote this new graph by $G(V, E')$. Since by assumption u_2 is in the geodesic path from v_1 to v_2 and u_1 is not, $G(V, E')$ again is a connected d-regular tree with boundary. Obviously $\mu(G') = \mu(G)$.

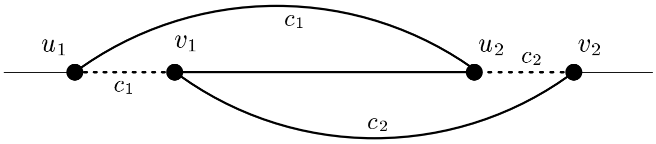

FIGURE 5. Rearrangement step.

Lemma 5 (rearrangement of edges, [5, lemma 5]). *Construct a d-regular tree G' with boundary as described above. Then $\nu(G') \leq \nu(G)$ whenever $f(v_1) \geq f(u_2)$, $f(v_2) \geq f(u_1)$ and $c_1 \leq c_2$. $\nu(G') < \nu(G)$ if and only if one of these three inequalities is strict.*

The normal derivative of f at the boundary edge $e_j = (v_j, u_j) \in \partial E$, $v_j \in V_0$, of length $c_j = c_{e_j}$ is $f(v_j)/c_j$. The "average" normal derivative of n boundary edges is given by $\sum_{j=1}^{n} f(v_j) / \sum_{j=1}^{n} c_j$. We replace each of these n edges e_j by edges $\bar{e}_j$ of length $\bar{c}_j$, where each $\bar{c}_j$ is given by

$$\bar{c}_j = f(v_j) \frac{\sum_{i=1}^{n} c_i}{\sum_{i=1}^{n} f(v_i)}. \tag{6}$$

Then the normal derivative is the same for all these boundary edges. It is clear that such an edge $\bar{e}_j$ might become longer than 1. Then we replace all the edges e_j by edges $e_j(\varepsilon)$ of lengths $c_j(\varepsilon) = (1 - \varepsilon)c_j + \varepsilon\, \bar{c}_j$, where $\varepsilon \in (0, 1]$. Make ε as great as possible, i.e. (either) one edge $e_j(\varepsilon)$ has length $c_j(\varepsilon) = 1$ or $\varepsilon = 1$. Denote the resulting graph by $G(\varepsilon)$.

Lemma 6 (perturbation of edges, [5, lemma 7]). *Construct a d-regular tree $G(\varepsilon)$ with boundary as described above. Then $\mu(G(\varepsilon)) = \mu(G)$ and $\nu(G(\varepsilon)) \leq \nu(G)$. Equality holds if and only if $\bar{c}_j = c_j$ in (6) for all j.*

Corollary 7. *Let $e_1, e_2 \in \partial E$ be two boundary edges with lengths $c_1, c_2 \in (0, 1]$. Let s_1, s_2 denote the normal derivatives at these edges. If $s_1 < s_2$ then we can decrease $\nu(G)$ by the above perturbation $G(\varepsilon)$ when we reduce the length of e_1 (where the normal derivative is "too small") and increase the length of e_2 (where f is "too steep".)*

Notice that we can apply this corollary only if $c_2 < 1$ since otherwise we would obtain a tree with boundary edge with length greater than 1.

Proof of lemma 3. For (M1) see [4, theorem 4.4]. Now enumerate the vertices of G such that $v_0 = m$ and $i < j$ implies $f(v_i) \geq f(v_j)$. Define a well-ordering $\prec$ on the interior vertices of G, such that $v_i \prec v_j$ if and only if $i < j$. Then rearrange the interior edges as described above to make $\prec$ spiral-like on V_0. This rearrangement can be done stepwise: we make the first d vertices in the spiral-like ordering adjacent to the root m. Then connect the next $d-1$ vertices to v_2, the next $d-1$ vertices to v_3 and so on (for details see proof of lemma 6 in [5]). It easily follows that $\nu(G)$ is decreased by this rearrangement if (M3) does not hold for the original tree G (lemma 5). By rearranging the boundary edges such that longer boundary edges are incident to interior vertices with greater values of $f(v)$ we arrive at (M2) (for details see lemmata 6 and 8 in [5]). This rearrangements make it possible to modify the ordering $\prec$ on ∂V so that (S3) holds. Lemma 6 implies that (M4) must hold if G has the Faber-Krahn property ([5, lemma 7]). $\square$

Balanced branches are important for our onion-shaped trees. It is easy to compute the eigenfunction f on these subgraphs by a straightforward calculation (see [4, 5]) using

$$\Delta f(v) = \sum_{e_i=(v,u_i)\in E} \frac{1}{c_{e_i}}(f(v) - f(u_i)) = \nu\, f(v). \tag{7}$$

Notice that $\frac{1}{c_{e_i}} = 1$ for all but the boundary edges.

Lemma 8. *Let $\{(v_0,v_1),(v_1,v_2),\dots,(v_{n-1},v_n)\}$ be a geodesic path in G with $v_0 \in \partial V$ and $v_i \in V_0$, $i = 1,\dots,n$. Let c denote the length of the boundary edge (v_0,v_1). If $\mathrm{Br}(v_j,v_{j-1})$, $j = 1,\dots,n$, are balanced branches then*

$$\begin{aligned} f(v_2) &= ((d-1) + (1-\nu)c)\, f(v_1)/c \\ f(v_j) &= (d-\nu) f(v_{j-1}) - (d-1) f(v_{j-2}), \qquad j = 3,\dots,n. \end{aligned} \tag{8}$$

The above lemma shows that we can express $f(v_j)$ by

$$f(v_j) = s\,(\alpha_j(d,\nu) + \beta_j(d,\nu)\, c) \tag{9}$$

where s denotes the normal derivative at the boundary edge (v_1,v_0), i.e. $f(v_1)/c$. The coefficients α_j and β_j are polynomials which are given by the recursion

$$\begin{aligned} &\alpha_1 = 0, \quad \alpha_2 = d-1 \quad \text{and} \quad \alpha_i = (d-\nu)\,\alpha_{i-1} - (d-1)\,\alpha_{j-2}, \\ &\beta_1 = 1, \quad \beta_2 = 1-\nu \quad \text{and} \quad \beta_i = (d-\nu)\,\beta_{i-1} - (d-1)\,\beta_{i-2}. \end{aligned} \tag{10}$$

Lemma 9. *Let $d \geq 3$ be fixed. Then*

(i) $\beta_k(d,\nu)$ is a polynomial in ν of degree $k-1$, with $\beta_k(d,0) = 1$ for all k and $\beta_k(d,\nu) = (-\nu)^{k-1} + O(\nu^{k-2})$ for $\nu \to \infty$.
(ii) $\beta_k(d,\nu)$ has $k-1$ distinct real roots $0 < \nu_1 < \nu_2 < \ldots < \nu_{k-1}$.
(iii) For $k \geq 3$ the roots of $\beta_k(d,\nu)$ and $\beta_{k-1}(d,\nu)$ are interlaced, i.e. for each of the (open) intervals (v_j, v_{j+1}) there exists exactly one root ν of $\beta_{k-1}(d,\nu)$ with $\nu_j < \nu < \nu_{j+1}$.

Proof. (i) follows immediately from recurrence (10). The other statements are trivial for β_1 and β_2. Now assume that the proposition holds for $k \geq 2$. Then by recurrence (10) we find for each root ν_j of β_k, $\beta_{k+1}(d,\nu_j) = -(d-1)\beta_{k-1}(d,\nu_j)$. By (i) and (iii) we have $\operatorname{sign}\beta_{k-1}(d,\nu_j) = -\operatorname{sign}\beta_{k-1}(d,\nu_{j+1}) \neq 0$ and thus there is a root of β_{k+1} in each interval (ν_j,ν_{j+1}). Moreover since there is no root of β_{k-1} less than ν_1 or greater than ν_{k-1} it follows immediately from (i) that there is a root of β_{k+1} in the intervals $(0,\nu_1)$ and (ν_{k-1},∞), respectively. Thus (ii) and (iii) follows by induction. $\square$

We get the following consequence of (iii).

Corollary 10. *Let $d \geq 3$ be fixed. Then $\beta_j(d,\nu) > 0$ for all $j \leq k$ if and only if ν is less than the smallest root of β_k.*

Lemma 11. *For $k \geq 1$ we find*

$$\alpha_{k+1}(d,\nu) = (d-1)\,(\alpha_k(d,\nu) + \beta_k(d,\nu)) \tag{11}$$

$$\beta_{k+1}(d,\nu) = (1-\nu)\,\beta_k(d,\nu) - \nu\,\alpha_k(d,\nu) \tag{12}$$

Proof. We have $\alpha_2 = (d-1)$ and $\alpha_3 = (d-1)\,(d-\nu) = (d-1)\,(\alpha_2+\beta_2)$. Furthermore $\beta_2 = (1-v)$ and $\beta_3 = (d-\nu)(1-\nu) - (d-1) = (1-\nu)(1-\nu) - \nu\,(d-1) = (1-\nu)\,\beta_2 - \nu\,\alpha_2$. Thus the proposition follows by induction from recursion (10). □

As an immediate consequence of this lemma we find (notice that $\beta_1(d,\nu) = 1$)

Corollary 12. *For $k \geq 2$, $\alpha_k(d,\nu) > 0$ if $\beta_j(d,\nu) \geq 0$ for all $j \leq k-1$.*

Notice that $f(v)$ only depends on $h(v)$ in a balanced branch, whether G has the Faber-Krahn property or not. Otherwise there would exist two independent eigenfunctions on G and thus $\nu(G)$ would not be simple, a contradiction. Therefore (8)–(10) hold whenever $\mathrm{Br}(v_j, v_{j-1})$ are balanced branches, even if the chosen root for G is not a maximum of f or G does not have the Faber-Krahn property.

4. Proof of the theorem

Again we assume that $G = G(V_0 \cup \partial V, E_0 \cup \partial E)$ has the Faber-Krahn property and f is a nonnegative eigenfunction to the first Dirichlet eigenvalue $\nu(G)$ with maximum m as the root of tree G. k always denotes an integer ≥ 1.

We show that a connection exists between the sign of the polynomials $\beta_j(d,\nu)$ at $\nu(G)$ and the existence of balanced or unbalanced branches of integer or non-integer length.

Lemma 13. *Let $\mathrm{Br}(v,u_1)$ and $\mathrm{Br}(v,u_2)$ be two balanced branches of lengths ℓ_1 and ℓ_2, respectively, $u_1 \neq u_2$, with $k-1 < \ell_1 \leq \ell_2 < k$ for a $k \in \mathbb{N}$. If $\beta_k(d,\nu(G)) \neq 0$, then $\ell_1 = \ell_2$.*

Proof. By (M4) and (9) we find $f(v) = s\,(\alpha_k(d,\nu(G)) + \beta_k(d,\nu(G))\,(\ell_i - k + 1))$, for $i = 1,2$. Consequently $\beta_k(d,\nu(G))\,(\ell_1-\ell_2) = 0$ and thus $\ell_1 = \ell_2$. □

Lemma 14. *Let $\mathrm{Br}(v,u_1)$ and $\mathrm{Br}(v,u_2)$ be to two balanced branches of length ℓ, $u_1 \neq u_2$, with $k-1 < \ell < k$ for a $k \in \mathbb{N}$. If $\beta_j(d,\nu(G)) \geq 0$ for every $j \leq k-1$, then $\beta_k(d,\nu(G)) \geq 0$.*

Proof. Replace the boundary edges of length $c = \ell - k + 1$ in $\mathrm{Br}(v,u_1)$ by boundary edges of length $c_1 = c - \varepsilon$ and those in $\mathrm{Br}(v,u_2)$ by edges of length $c_2 = c + \varepsilon$, for a sufficiently small $\varepsilon > 0$. Denote the resulting graph (which might not have the Faber-Krahn property) by G_ε. Then $\mu(G_\varepsilon) = \mu(G)$ and $\nu(G_\varepsilon) \geq \nu(G)$. Let s_1 and s_2 denote the respective normal derivatives of the positive eigenfunction f_ε at these modified boundary edges. Then we find by (9), $f_\varepsilon(v) = s_i\,(\alpha_k(d,\nu(G_\varepsilon)) + \beta_k(d,\nu(G_\varepsilon))\,c_i)$, $i = 1,2$.

Now suppose $\beta_k(d,\nu(G)) < 0$. Then by the continuity of $\nu(G)$ (proposition 1), $\beta_k(d,\nu(G_\varepsilon)) < 0$ for all sufficiently small $\varepsilon > 0$. Notice that by assumption and corollary 12, $\alpha_j(d,\nu(G)) > 0$ for every $j \leq k$. Then we find

$$s_1 = s_2 \cdot \frac{\alpha_k(d,\nu(G_\varepsilon)) + \beta_k(d,\nu(G_\varepsilon))\,(c+\varepsilon)}{\alpha_k(d,\nu(G_\varepsilon)) + \beta_k(d,\nu(G_\varepsilon))\,(c-\varepsilon)} < s_2 \tag{13}$$

where the inequality holds if and only if $\beta_k(d, \nu(G_\varepsilon)) < 0$. By corollary 7 we can decrease the first Dirichlet eigenvalue of G_ε by constructing a new graph $G_{\varepsilon'}$ by making the boundary edges of length $c_1 = c - \varepsilon$ shorter to get length $c'_1 = c - \varepsilon' < c_1$ and the boundary edges of length $c_2 = c + \varepsilon$ longer to get length $c'_2 = c + \varepsilon' > c_2$ (we assume without loss that $\varepsilon' > \varepsilon$ is sufficiently small). Notice that for the new eigenfunction $f_{\varepsilon'}$, $s'_1/s'_2 < s_1/s_2 < 1$. Thus we can make ε' even a little bit larger and we can set $\varepsilon' = 2\varepsilon$ (for sufficiently small ε). Hence and by the Faber-Krahn property of G we find $\nu(G_\varepsilon) > \nu(G_{2\varepsilon}) \geq \nu(G)$. These inequalities hold for every $\varepsilon_i = \varepsilon/2^i$ and we arrive at $\nu(G_{\varepsilon_i}) > \nu(G_{\varepsilon_0}) > \nu(G)$, i.e. $\nu(G_{\varepsilon_i})$ cannot converge to $\nu(G)$, a contradiction to the continuity of ν. Hence $s_1 \geq s_2$ and $\beta_k(d, \nu(G_\varepsilon)) \geq 0$ for every sufficiently small $\varepsilon > 0$. Thus $\beta_k(d, \nu(G)) \geq 0$ as claimed. □

Lemma 15. *Let* $\mathrm{Br}(v, u_1)$ *and* $\mathrm{Br}(v, u_2)$ *be two balanced branches of lengths* ℓ_1 *and* ℓ_2*, respectively,* $u_1 \neq u_2$*, with* $k - 1 = \ell_1 < \ell_2 \leq k$ *for a* $k \in \mathbb{N}$*,* $k \geq 2$*. If* $\beta_j(d, \nu(G)) \geq 0$ *for every* $j \leq k - 1$*, then* $\beta_k(d, \nu(G)) \leq 0$*.*

Proof. Construct a new regular tree with boundary G_ε for a sufficiently small $\varepsilon > 0$ by replacing the boundary edges of length $c = \ell_2 - k$ in $\mathrm{Br}(v, u_2)$ by boundary edges of length $c - \varepsilon$ and by including new boundary edges at the boundary vertices in $\mathrm{Br}(v, u_1)$ (which then become interior vertices) of length ε. We then have $\mu(G_\varepsilon) = \mu(G)$ and $\nu(G_\varepsilon) \geq \nu(G)$. Let s_1 and s_2 denote the respective normal derivatives of the positive eigenfunction f_ε at these modified boundary edges. Now suppose $\beta_k(d, \nu(G)) > 0$. Then by the continuity of $\nu(G)$, $\beta_k(d, \nu(G_\varepsilon)) > 0$ for all sufficiently small $\varepsilon > 0$. Then we find for sufficiently small $\varepsilon > 0$

$$s_1 = s_2 \cdot \frac{\alpha_k(d, \nu(G_\varepsilon)) + \beta_k(d, \nu(G_\varepsilon))\,(c - \varepsilon)}{\alpha_k(d, \nu(G_\varepsilon)) + \beta_k(d, \nu(G_\varepsilon))\,\varepsilon} > s_2 \tag{14}$$

i.e. we can decrease the first Dirichlet eigenvalue by replacing ε by a sufficiently small $\varepsilon' > \varepsilon > 0$. Therefore analogously to the proof of lemma 14 we find $\beta_k(d, \nu(G)) \leq 0$ as claimed. □

Lemma 16. *Let* $\mathrm{Br}(v, u_1)$ *and* $\mathrm{Br}(v, u_2)$ *be two balanced branches of lengths* ℓ_1 *and* ℓ_2*, respectively,* $u_1 \neq u_2$*, with* $k - 1 < \ell_1 < \ell_2 = k$ *for a* $k \in \mathbb{N}$*. Then* $\beta_k(d, \nu(G)) \leq 0$*.*

Proof. By (9) we have $f(v) = s_1\,(\alpha_k + \beta_k\,(\ell_1 - k + 1)) = s_2\,(\alpha_k + \beta_k\,(\ell_2 - k + 1))$. By corollary 7 we must have $s_1 \leq s_2$ and thus $\beta_k\,\ell_1 \geq \beta_k\,\ell_2$ which holds if and only if $\beta_k \leq 0$. □

Lemma 17. *Let* $\mathrm{Br}(v, w)$ *be a branch such that* $k < \mathrm{dist}(v, u_0) < k + 1$*,* $k \in \mathbb{N}$*, for each boundary vertex* u_0 *in* $\mathrm{Br}(v, w)$*. If* $\nu(G)$ *is not the smallest root of any* $\beta_j(d, \nu)$ *for* $j \leq k$*, then* $\mathrm{Br}(v, w)$ *is balanced and* $\beta_j(d, \nu) > 0$ *for all* $j = 1, \ldots, k$*. Moreover* $\nu(G)$ *is less than the smallest root of* $\beta_k(d, \nu)$*.*

Proof. $\mathrm{Br}(v, w)$ is balanced for $k = 1$ by (M4) and $\beta_1(d, \nu)$ has no roots. Now assume the proposition holds for $k - 1 \geq 1$, i.e. all subbranches $\mathrm{Br}(w, u_i)$ are balanced and $\beta_j(d, \nu(G)) > 0$ for $j \leq k - 1$. By corollary 10 there is no root of $\beta_k(d, \nu)$ because the smallest one has been excluded by assumption. Then by lemma 13 all these subbranches have the same length, i.e. $\mathrm{Br}(v, w)$ is balanced and by lemma 14, $\beta_k(d, \nu(G)) > 0$. Thus the proposition follows by induction. The last statement is an immediate consequence of corollary 10. □

Lemma 18. *If $\nu(G)$ is not the smallest root of any $\beta_j(d,\nu)$ for $j \leq k+1$, then there are no balanced branches* $\mathrm{Br}(v,u_1)$ *and* $\mathrm{Br}(v,u_2)$ *of lengths ℓ_1 and ℓ_2, respectively, such that $k-1 < \ell_1 < k < \ell_2 < k+1$ for a $k \in \mathbb{N}$ (see figure 6).*

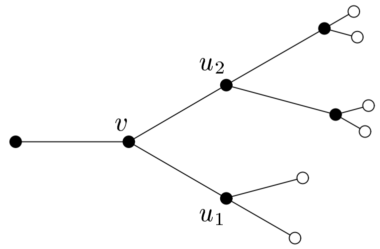

FIGURE 6. Unbalanced branch.

Proof. Suppose two such branches exist. By (M4), eq. (9) and lemma 11, $f(v) = \alpha_k + \beta_k\, c_1 = \alpha_{k+1} + \beta_{k+1}\, c_2 = (d-1)\alpha_k + (d-1)\beta_k + ((1-\nu)\beta_k - \nu\alpha_k)\, c_2$, where $c_1 = \ell_1 - k + 1$ and $c_2 = \ell_2 - k$ denote the lengths of the boundary edges. Thus we find

$$(d-2-\nu\, c_2)\,\alpha_k + (d-1-c_1+(1-\nu)\,c_2)\,\beta_k = 0. \tag{15}$$

Notice that by lemma 17, $\beta_j(d,\nu(G)) > 0$ for all $j \leq k$. Hence for $k \geq 2$ we have $\nu < 1$. Moreover $c_i \in (0,1)$ and $d \geq 3$. Thus the left hand side of eq. (15) is greater than 0, a contradiction.

Now suppose $k = 1$. Since $\alpha_1 = 0$ and $\beta_1 = 1$, the left hand side of eq. (15) reduces to $d-1-c_1+(1-\nu)\,c_2 = 0$. Consequently $\nu(G) = (d-c_1+c_2-1)/c_2 > (d-2)/c_2 + 1 > d-1 = \nu(B_d(z,1.5))$, where z is the mid point of some edge. Consequently $B_d(z,1.5) \not\subseteq G$ and by (M2) we can choose vertex v as the root m of G. By lemma 13 all branches at v are balanced and have either length $\ell_1 \in (0,1)$ or length $\ell_2 \in (1,2)$. If there are two branches of length ℓ_2 then by lemma 14 $\beta_2(d,\nu(G)) > 0$ and thus $\nu(G)$ would be less than 1. Therefore $G \subseteq B_d(z,1.5)$ consists of (exactly) two interior vertices v and u_2 (u_1 is then a boundary vertex). Since without loss $c_1 \geq c_2$ (otherwise change the rôle of u_2 and v) we have by (M4), $f(v) \geq f(u_2)$. Now construct a new graph G_ε by replacing the boundary edges at u_2 with boundary of edges of length $c_2 - \varepsilon$ and the boundary edges at v with boundary edges of length $c_1 + \varepsilon$, for sufficiently small $\varepsilon > 0$. Obviously $\mu(G_\varepsilon) = \mu(G)$. Further define a function f_ε on G_ε by $f_\varepsilon(v) = f(v) + \varepsilon$ and $f_\varepsilon(u_2) = f(u_2) - \varepsilon$. Then using the Rayleigh quotient we find for $\nu(G) > d-1 \geq 2$

$$\frac{\langle \Delta(G_\varepsilon)\, f_\varepsilon, f_\varepsilon\rangle}{\langle f_\varepsilon, f_\varepsilon\rangle} = \frac{\langle \Delta(G)\, f, f\rangle + 4\varepsilon(f(v)-f(u_2)) + 4\varepsilon^2}{\langle f, f\rangle + 2\varepsilon(f(v)-f(u_2)) + 2\varepsilon^2} < \frac{\langle \Delta(G)\, f, f\rangle}{\langle f, f\rangle} = \nu(G)$$

where the inequality holds since $(f(v)-f(u_2)) + \varepsilon > 0$. Thus G cannot have the Faber-Krahn property, a contradiction. $\square$

Lemma 19. *Let* $\mathrm{Br}(v,w)$ *be a branch of length ℓ with $k \leq \ell \leq k+1$ for a $k \in \mathbb{N}$, such that there exists a boundary edge of length $c < 1$. Assume that $\nu(G)$ is not the root of any $\beta_j(d,\nu)$ for $j \leq k$. Then* $\mathrm{Br}(v,w)$ *is balanced if and only if $\nu(G)$ is less than the smallest root of $\beta_k(d,\nu)$.*

Proof. If $\mathrm{Br}(v,w)$ is balanced then $k<\ell<k+1$ and the result follows immediately from lemma 17.

Now assume $\nu(G)$ is less than the smallest root of $\beta_k(d,\nu)$. For $k=1$, $\mathrm{Br}(v,w)$ is balanced by (M4) and $\beta_1(d,\nu)=1$. Assume the proposition holds for $k-1$. Then every subbranch $\mathrm{Br}(w,v_i)$ of $\mathrm{Br}(v,w)$ has length $\ell_i \le k$ and is balanced. Since $\beta_j(d,\nu(G))>0$ for $j\le k$, $m\notin \mathrm{Br}(v,w)\setminus\{v\}$ as a consequence of corollary 12. Thus by (M2), $\ell_i>k-2$ and thus by lemma 18, $k-1\le \ell_i\le k$. Therefore we have by lemmata 15, 16 and 13 that all subbranches have the same length $\ell_i=\ell-1$. Thus $\mathrm{Br}(v,w)$ is balanced and the proposition follows by induction. □

Lemma 20. *Let* $\mathrm{Br}(v,w)$ *be a branch of length* ℓ *with* $k<\ell\le k+1$ *for a* $k\in\mathbb{N}$. *If* $\beta_j(d,\nu(G))>0$ *for all* $j\le k$, *then* $\mathrm{Br}(v,w)$ *is balanced.*

Proof. Analogously to the second part of the proof of lemma 19 by induction. □

Lemma 21. *Let k be the (smallest) integer such that* $\beta_j(d,\nu(G))>0$ *for all* $j\le k$ *but* $\beta_{k+1}(d,\nu(G))\le 0$. *Then there are no two branches* $\mathrm{Br}(v_1,u_1)$ *and* $\mathrm{Br}(v_2,u_2)$ *of lengths* ℓ_1 *and* ℓ_2, *respectively,* $v_1\ne v_2$, *with* $k<\ell_1\le\ell_2<k+1$.

Proof. Suppose there would be two such branches $\mathrm{Br}(v_1,u_1)$ and $\mathrm{Br}(v_2,u_2)$. Let $\mathrm{Br}(w_1,v_1)\supset\mathrm{Br}(v_1,u_1)$ and $\mathrm{Br}(w_2,v_2)\supset\mathrm{Br}(v_2,u_2)$. By lemma 19 all subbranches of $\mathrm{Br}(w_1,v_1)$ and $\mathrm{Br}(w_2,v_2)$ are balanced. Since $\beta_{k+1}(d,\nu(G))\le 0$ we find by (M4) and (9), $f(v_1)\ge f(v_2)$. But by corollary 4 we have $f(v_1)\le f(v_2)$. Thus $f(v_1)=f(v_2)$, $\ell_1=\ell_2$ and consequently $f(u_1)=f(u_2)$. Let $\mathrm{Br}(v_1,t_1)$ and $\mathrm{Br}(v_2,t_2)$, $t_j\ne u_j$, be subbranches. Without loss we assume $w_1\prec w_2$ and furthermore by the properties of the spiral-like ordering we find $t_1\prec u_1\prec u_2\prec t_2$ (otherwise change the rôle of u_i and t_i). By (M3), $f(w_1)\ge f(w_2)$ and $f(x_1)\ge f(x_2)$ for all $x_1\in\mathrm{Br}(v_1,t_1)$ and $x_2\in\mathrm{Br}(v_2,t_2)$ with $\mathrm{dist}(x_1,w_1)=\mathrm{dist}(x_2,w_2)$. This holds for every such t_1 and t_2. Applying (7) we find $(d-\nu(G))(f(v_1)-f(v_2))=f(w_1)-f(w_2)+f(u_1)-f(u_2)+\sum_{t_1\sim v_1}f(t_1)-\sum_{t_2\sim v_2}f(t_2)=0$. Thus $f(w_1)=f(w_2)$ and $f(t_1)=f(t_2)$. But since $\mathrm{Br}(v_i,t_i)$ are balanced we find by a straightforward computation (in the reverse order of lemma 8) using (7) that $f(x_1)=f(x_2)$ for all $x_1\in\mathrm{Br}(v_1,t_1)$ and $x_2\in\mathrm{Br}(v_2,t_2)$ with $\mathrm{dist}(x_1,w_1)=\mathrm{dist}(x_2,w_2)$. Consequently all subbranches at v_1 and v_2 must have the same length and by lemma 14 $\beta_{k+1}(d,\nu(G))\ge 0$ and thus $\beta_{k+1}(d,\nu(G))=0$. Thus the branches $\mathrm{Br}(w_1,v_1)$ and $\mathrm{Br}(w_2,v_2)$ are balanced branches of length $k+1<\ell<k+2$.

Now by lemma 9, $\beta_{k+2}(d,\nu(G))<0$. Thus $w_1\ne w_2$ by lemma 14. But by repeating the same computation with $\mathrm{Br}(w_1,v_1)$ and $\mathrm{Br}(w_2,v_2)$ instead of $\mathrm{Br}(v_1,u_1)$ and $\mathrm{Br}(v_2,u_2)$ we find $\beta_{k+2}(d,\nu(G))\ge 0$, a contradiction. □

Lemma 22. *If* $\nu(G)$ *is not the smallest root of any* $\beta_j(d,\nu)$, *then there exists at most one unbalanced branch of length* ℓ *with* $k-1<\ell\le k$ *for all* $k\in\mathbb{N}$, $k\le \max_{v\in V}h(v)$.

Proof. By the spiral-like ordering (S2 and S3) we find for two vertices $w_1\prec w_2$ with same height $h(w_i)$, that all branches at w_1 cannot be shorter than the branches at w_2. Since there is at most one branch that contains boundary edges of length <1 (lemma 21), the proposition follows. □

In spite of the fact that we need no restrictions on the possible values for k in the above lemmata (except the diameter of the tree), only a few special cases can occur, i.e. the smallest root of a $\beta_j(d,\nu)$ is less than $\nu(G)$ in almost all cases. For the polynomials $\beta_k(d,\nu)$ we find by recursion (10)

$$\begin{array}{rcl} \beta_1(d,\nu) &=& 1 \\ \beta_2(d,\nu) &=& 1-\nu \\ \beta_3(d,\nu) &=& 1-(1+d)\,\nu+\nu^2 \\ \beta_4(d,\nu) &=& 1-(2+d^2)\,\nu+(1+2d)\,\nu^2-\nu^3 \end{array} \tag{16}$$

The smallest roots and bounds for $\nu(G)$ (by proposition 2) are

$$\begin{array}{llccc} & & d=3 & d=4 & d\geq 5 \\ & \nu(G) & >0.171 & >0.535 & >1 \\ & \beta_1 & - & - & - \\ \text{smallest} & \beta_2 & 1 & 1 & 1 \\ \text{root of} & \beta_3 & 2-\sqrt{3} & \approx 0.209 & \\ & \beta_4 & \approx 0.097 & & \end{array} \tag{17}$$

The balls $B_d(z, j+\frac{1}{2})$, where z is the midpoint of some edge, are of special importance for the theorem. Notice that by symmetry the eigenfunction f must be constant on the edge containing z. Thus we have $f(v_j)=f(v_{j+1})$ in recursion (8) and consequently we find by a straightforward computation for the first Dirichlet eigenvalue

$$\nu(B_4(z,4.5))=1, \quad \nu(B_3(z,2.5))=1 \quad \text{and} \quad \nu(B_3(z,9.5))=2-\sqrt{3}. \tag{18}$$

Notice that for a ball $B_d(z, j+1/2)$ there exists a root m and $d-1$ balanced branches of length j and one balanced branch of length $j+1$ at m.

Now we are ready to prove the Faber-Krahn theorem for regular trees.

Proof of the necessity condition. Assume G has the Faber-Krahn property. We have to show that G is onion shaped. We first assume that $\nu(G)$ is not the smallest root of any $\beta_j(d,\nu)$.

(O1) is just (M1). (O2) is an immediate consequence of (M2) and lemma 18. (O3) follows from lemmata 19 and 21. (O4) follows from lemma 22 since a branch cannot be balanced if it has an unbalanced subbranch.

Now assume that there are at least two interior vertices (otherwise we have case (F0)). We have to show that one of the conditions (F1), (F2) or (F3) holds.

If $\beta_2(d,\nu(G))<0$ then all branches of length $\ell\in(1,2]$ are balanced by lemma 20 since $\beta_1(d,\nu)=1>0$. Moreover by lemma 21 there is at most one balanced branch of length $\ell\in(1,2)$. By (17) and (18), $\beta_2(d,\nu(G))<0$ if $d\geq 5$. If $d=4$ then we must have $\nu(G)>\nu(B_4(z,4.5))$. If $G\not\subset B_4(z,4.5)$ then by the basic properties of the first Dirichlet eigenvalue (proposition 1) $B_4(z,4.5)\not\subseteq G$, but $B_4(m,4)\subseteq G$ by the onion shape of G. Thus there must be at least two balanced branches of length $\ell\in(1,2)$, a contradiction to lemma 21 and hence G must satisfy (F1). Analogously for $d=3$, $G\subseteq B_3(z,2.5)$. Thus condition (F1) holds.

If $\beta_3(d,\nu(G))<0$ but $\beta_2(d,\nu(G))>0$, then again by lemma 20 all branches of length $\ell\in(2,3]$ are balanced and there is at most one balanced branch of length $\ell\in(2,3)$ by lemma 21. However this case only occurs if either $d=4$ and $\nu(G)<1$ and consequently $G\supseteq B_4(z,4.5)$. Or $d=3$ and $B_3(z,2.5)\subseteq G\subseteq B_3(z,9.5)$, i.e. condition (F2).

If $\beta_4(d,\nu(G))<0$ but $\beta_3(d,\nu(G))>0$ and $\beta_2(d,\nu(G))>0$ then analogously (F3) must hold. By (17) and (18) no other cases are possible.

Now assume $\beta_2(d, \nu(G)) = 0$, i.e. $\nu(G) = 1$. Let $d = 3$. If $G \neq B_3(z, 2.5)$ then neither $G \not\subset B_3(z, 2.5)$ nor $G \not\supset B_3(z, 2.5)$, by proposition 1. But then we must have at least to branches of length $\ell \in (1, 2)$, a contradition to lemma 21. Analogously the propostion follows for the remaining cases $\beta_2(d, \nu(G)) = 0$ and $d = 1$ and $\beta_3(d, \nu(G)) = 0$ and $d = 3$. □

Proof of the sufficiency condition. The set of all connected d-regular trees with boundary of fixed finite volume and containing a fixed vertex v is compact. Thus by the continuity of $\nu(G)$, a tree G^* that minimizes ν (i.e. with the Faber-Krahn property) exists.

Now let $G(V_0 \cup \partial V, E_0 \cup \partial E)$ and $G^*(V_0^* \cup \partial V^*, E_0^* \cup \partial E^*)$ be onion shaped d-regular trees with boundary that fulfill one of the properties (F0), (F1), (F2) or (F3), with $\mu(G) = \mu(G^*)$. By these properties the number of interior and boundary vertices and edges is uniquely defined for a given volume $\mu(G)$. Therefore there exists a homomorphism $H : G \to G^*$ with $H(V_0) = V_0^*$ and $H(\partial V) = \partial V^*$ and for all $(v, w) \in E$ the edge $(H(v), H(w)) \in E^*$ has the same length as (v, w). If G and G^* have the Faber-Krahn property, uniqueness up to homomorphism follows. □

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