A Faber-Krahn-type inequality for regular trees

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Abstract

We show a Faber-Krahn-type inequality for regular trees with boundary.

1. Introduction

The eigenvectors of the Laplacian Δ on graphs have received little attention compared to the spectrum of this operator (see e.g. [6, 9, 10]) or the eigenfunctions of the "classical" Laplacian differential operator on Riemannian manifolds (e.g. [1, 2]). Recently these eigenvectors seem to become more important. Grover [7] has discovered that the cost function of a number of well-studied combinatorial optimisation problems, e.g. the *travelling salesman problem*, are eigenvectors of the Laplacian of certain graphs. Thus global properties of such eigenvectors are of interest.

In the last years some results for the Laplacian on manifolds have been shown to hold also for the graph Laplacian, e.g. Courant's nodal domain theorem ([3, 5]) or Cheeger's inequality ([4]). In [5] Friedman described the idea of a "graph with boundary" (see below). With this concept he was able to formulate Dirichlet and Neumann eigenvalue problems. He also conjectured another "classical" result for manifolds, the Faber-Krahn theorem, for regular bounded trees with boundary. The Faber-Krahn theorem states that among all bounded domains $D \subset \mathbb{R}^n$ with fixed volume, a ball has lowest first Dirichlet eigenvalue.

In this paper we want to show such a result for trees. We give restrictive conditions for trees with boundary where the first Dirichlet eigenvalue is minimized for a given "volume". Amazingly Friedman's conjecture is false, i.e. in general these trees are not "balls". But we will show that these are similar to "balls".

2. Statement of the Result

Let $G = (V, E)$ be an undirected (weighted) graph, with weights $\frac{1}{c_e} > 0$ for each $e \in E$. The *Laplacian* of G is the matrix

$$\Delta = \Delta(G) = D(G) - A(G)$$

where $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix whose entries are the sums of the weights of the edges at the vertices of G, i.e. $D_{v,v} = \sum_{e=(v,u)\in E} \frac{1}{c_e}$. The associated Rayleigh quotient on real-valued functions f on V is the fraction

$$\mathcal{R}_G(f) = \frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} = \frac{\sum_{(u,v)\in E} \frac{1}{c_e}(f(u) - f(v))^2}{\sum_{v\in V} (f(v))^2}$$

Notice that in opposite to the Laplacian differential operator on manifolds, $\Delta(G)$ is defined as a positive operator.

The *geometric realization* of G is the metric space $\mathcal{G}$ consisting of V and arcs of length c_e glued between u and v for every edge $e = (u, v) \in E$. We define two measures on $\mathcal{G}$ (and G). Let $\mu_1(\mathcal{G}) = |V|$ be the number of vertices of G and $\mu_2(\mathcal{G}) = \sum_{e\in E} c_e$, i.e. the Lebesgue measure of $\mathcal{G}$. Now let $\mathcal{S}$ denote the set of all continuous functions on $\mathcal{G}$ which are differentiable on $\mathcal{G} \setminus V$. We introduce a (Laplacian) operator $\Delta_\mathcal{G}$ on $\mathcal{G}$ by the Rayleigh quotient

$$\mathcal{R}_\mathcal{G}(f) = \frac{\int_\mathcal{G} |\nabla f|^2 d\mu_2}{\int_\mathcal{G} f^2 d\mu_1}, \qquad f \in \mathcal{S}$$

The operator $\Delta_\mathcal{G}$ is the continuous version of the Laplacian Δ on G.

Proposition 1: (see [5])
The Rayleigh quotient $\mathcal{R}_\mathcal{G}(f)$ is minimized at, and only at, edgewise linear functions $f \in \mathcal{S}$, i.e. those functions whose restrictions to each edge are linear.
The eigenvalues and eigenfunctions of $\Delta_\mathcal{G}$ exist and are those of Δ (i.e. the restrictions of the $\Delta_\mathcal{G}$ eigenfunctions to V are the Laplacian eigenvectors).

On $\mathcal{G}$ we can avoid the problems that arise from the discreteness of our situation. Now the *(proper) nodal domains* of an eigenfunction f of $\Delta_\mathcal{G}$ are the components of the complement of $f^{-1}(0)$, i.e. of the nodal set of f. Thus analogously to the classical situation (see [2]) f vanishes on the "boundary" of each nodal domain.

It makes sense to introduce the *Dirichlet eigenvalue problem* for graphs with boundary. A *graph with boundary* is a graph $G(V_0 \cup \partial V, E_0 \cup \partial E)$ where each vertex in ∂V (*boundary vertex*) has degree 1 (i.e. it is the endpoint of one edge not necessarily of length 1) and each vertex in V_0 (*interior vertex*) has degree greater than or equal to 2. Each edge $e \in E_0$ (*interior edge*) joins two interior vertices, each edge $e \in \partial E$ (*boundary edge*) connects an interior vertex with a boundary vertex. On such a graph we can define the "Dirichlet operator" by restricting f in the Rayleigh quotient $\mathcal{R}_\mathcal{G}(f)$ to those functions $f \in \mathcal{S}$ which vanish at all boundary vertices. Then the Dirichlet eigenvalue problem is to find the eigenvalues and eigenfunctions of this operator. Equivalently we can define this Laplacian operator on a graph with boundary by a linear operator that acts on the interior vertices of G only, i.e. on V_0:

$$\Delta_0 = D_0 - A_0$$

where A_0 is the adjacency matrix restricted to V_0 and where D_0 is the diagonal matrix whose entry corresponding to $v \in V_0$ is (note $E = E_0 \cup \partial E$)

$$(D_0)_{v,v} = \sum_{e=(v,u)\in E} \frac{1}{c_e}$$

Our goal is to find the eigenvalues and eigenvectors of this Laplacian.

If we now insert new vertices on each point in $\mathcal{G}$ where the eigenfunction f vanishes, then the closure of each nodal domain of f is the geometric realization of a graph. The restriction of f to this graph (i.e. the nodal domain) is an eigenfunction to the first Dirichlet eigenvalue of this graph.

Since there is no risk of confusion, we denote the Laplacian on a graph with boundary G simply by $\Delta = \Delta(G)$. We denote the lowest Dirichlet eigenvalue of G by $\nu(G)$. We then have the following properties of Δ.

Proposition 2: (see [5])
Let G be a graph with boundary.

(1) $\Delta(G)$ is a positive operator, i.e. $\nu(G) > 0$.

(2) An eigenfunction f to the eigenvalue $\nu(G)$ of $\Delta(G)$ is either positive or negative on all interior vertices of G.

(3) $\nu(G)$ is continuous as a function of G in the metric $\rho(G, G') = \mu_2(G - G') + \mu_2(G' - G)$.

(4) $\nu(G)$ is monotone in G, i.e. if $G \subset G'$ then $\nu(G) > \nu(G')$.

(5) $\nu(G)$ is a simple eigenvalue, if G is connected.

We refer the reader to [5] for the proofs and for more details.

In this paper we restrict our interest to regular trees with boundary. We get such a graph, when we take the geometric realization of an infinite d-regular tree and cut out a bounded region.

Definition 3:
A d-regular tree with boundary is a tree where all interior edges have length 1 (i.e. weight 1), all boundary edges length ≤ 1, and where all interior vertices have degree d and all boundary vertices degree 1. The set of interior vertices is not empty, i.e. $|V_0| \geq 1$.

We say a d-regular tree with boundary $G(V, E)$ fulfills the *Faber-Krahn-property*, if and only if $\nu(G) \leq \nu(G')$ for every d-regular tree with boundary G' with $\mu_2(G') = \mu_2(G)$.

A *ball* is d-regular tree with boundary with a center $c \in \mathcal{G}$, not necessarily a vertex, and a radius $r > 0$, such that $\mathrm{dist}(c, v_0) = r$ for all boundary vertices $v_0 \in \partial V$. $\mathrm{dist}(u, v)$ denotes the geodesic distance between $u, v \in \mathcal{G}$.

Every tree with the Faber-Krahn-property is "similar" to a ball.

Theorem 1:
Let $G(V_0 \cup \partial V, E_0 \cup \partial E)$ be a d-regular tree with the Faber-Krahn-property. Let f be a nonnegative eigenfunction of the first Dirichlet eigenvalue and m a maximum of f, i.e. $f(m) \geq f(v)$ for all $v \in V$. Then

(1) G is connected and

(2) $|dist(m, u_0) - dist(m, v_0)| \leq 1$, for all boundary vertices $u_0, v_0 \in \partial V$.

Now one might conjecture, that every tree with the Faber-Krahn-property is a ball centered at a vertex (see conjecture 4.3 in [5]). But this is not true in general.

Theorem 2:
If a ball $G(V_0 \cup \partial V, E_0 \cup \partial E)$ centered at a vertex has the Faber-Krahn-property, then all boundary vertices have length 1 or $|V_0| = 1$ or degree $d = 2$.

Theorem 3:
Let $G(V_0 \cup \partial V, E_0 \cup \partial E)$ be a d-regular tree with boundary with degree $d \geq 5$ which has the Faber-Krahn-property. Then there exists at most one vertex v, so that all boundary edges adjacent to v have length c, for a $c \in (0, 1)$. I.e. almost all boundary edges have length 1.

Figure 1 shows the geometric realization of a d-regular tree with boundary that has the Faber-Krahn-property. In this example $d = 6$ and $\mu_2(G) = 18$. There are 4 interior vertices.

3. Proof of the theorems

In the following we derive properties of trees with the Faber-Krahn property by rearrangements and small perturbations of d-regular graphs. We denote these

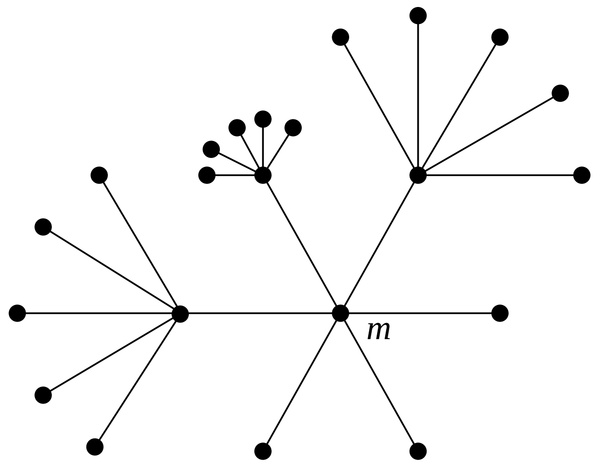

FIGURE 1: A 6-REGULAR TREE WITH THE FABER-KRAHN-PROPERTY

properties by (M1)–(M4). Notice that with this technique we only get necessary conditions for these types of trees.

We can restrict our interest to connected trees.

Proposition 4: (see [5], Theorem 4.4)
Every d-regular tree with boundary with the Faber-Krahn-property is connected.

Now we take an arbitrary d-regular tree with boundary. In certain situations it is possible to rearrange the edges of the trees so that $\nu(G)$ decreases.

Let $G(V_0 \cup \partial V, E_0 \cup \partial E)$ be a connected d-regular tree with boundary and f a nonnegative eigenfunction to the first Dirichlet eigenvalue $\nu(G)$. Let $(v_1, u_1), (v_2, u_2) \in E$ edges with lengths c_1 and c_2, respectively, so that u_2 is in the geodesic path from v_1 to v_2, but u_1 is not. Since G is a tree, $(v_1, v_2), (u_1, u_2) \notin E$. Thus we can replace edge (v_1, u_1) by edge (v_1, v_2) with length c_2 and edge (v_2, u_2) by edge (u_1, u_2) with length c_1. Denote this new graph by $G(V', E')$. Since by assumption u_2 is in the geodesic path from v_1 to v_2 and u_1 is not, $G(V, E')$ again is a connected d-regular tree with boundary (Figure 2 illustrates the situation). Obviously $\mu_2(G') = \mu_2(G)$.

Lemma 5:
Let $G(V, E)$ be a connected d-regular tree with boundary and f a nonnegative eigenfunction to the first Dirichlet eigenvalue $\nu(G)$. Construct a d-regular tree $G'(V, E')$ with boundary as described above.

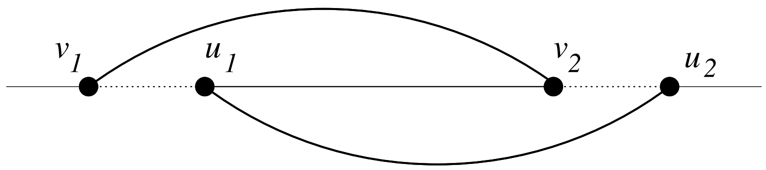

FIGURE 2

(1) Whenever $f(v_1) \geq f(u_2)$, $f(v_2) \geq f(u_1)$ and $c_1 \leq c_2$, then $\nu(G') \leq \nu(G)$.

(2) If one of these three inequalities is strict, then $\nu(G') < \nu(G)$.

PROOF: To verify (1) it remains to show that

$$\delta = \langle \Delta(G')f, f\rangle - \langle \Delta(G)f, f\rangle \leq 0 \tag{5.1}$$

Since we remove and insert two edges we have

$$\begin{aligned}\delta = &\left[\tfrac{1}{c_2}(f(v_1) - f(v_2))^2 + \tfrac{1}{c_1}(f(u_1) - f(u_2))^2\right] \\ &\quad - \left[\tfrac{1}{c_1}(f(v_1) - f(u_1))^2 + \tfrac{1}{c_2}(f(v_2) - f(u_2))^2\right] \\ = &\,(\underbrace{\tfrac{1}{c_1} - \tfrac{1}{c_2}}_{\geq 0})(\underbrace{f(u_2)^2 - f(v_1)^2}_{\leq 0}) + 2(\underbrace{\tfrac{1}{c_1}f(u_1) - \tfrac{1}{c_2}f(v_2)}_{\leq 0})(\underbrace{f(v_1) - f(u_2)}_{\geq 0}) \leq 0\end{aligned}$$

The third factor is nonpositive because $c_1 = 1$ or $f(u_1) = 0$ and $c_2 = 1$ or $f(v_2) = 0$ and $c_1 \leq c_2$ (If $c_1 < 1$ then u_1 is a boundary vertex.).

To prove (2) notice that $\nu(G') = \nu(G)$ if $\delta = 0$ in (5.1) and f is an eigenfunction to $\nu(G')$ on G', since $\nu(G')$ is simple (proposition 2). Therefore if $\nu(G') = \nu(G)$ we find

$$\begin{aligned}\nu(G)f(v_1) = \Delta(G)f(v_1) = &\sum_{w \sim v_1,\, w \neq u_1} \tfrac{1}{c_e}(f(v_1) - f(w)) + \tfrac{1}{c_1}(f(v_1) - f(u_1)) \\ = \nu(G')f(v_1) = \Delta(G')f(v_1) = &\sum_{w \sim v_1,\, v_1 \neq v_2} \tfrac{1}{c_e}(f(v_1) - f(w)) + \tfrac{1}{c_2}(f(v_1) - f(v_2))\end{aligned}$$

Thus

$$\tfrac{1}{c_1}(f(v_1) - f(u_1)) = \tfrac{1}{c_2}(f(v_1) - f(v_2))$$

and

$$f(v_1)(\underbrace{\tfrac{1}{c_1} - \tfrac{1}{c_2}}_{\geq 0}) = \underbrace{\tfrac{1}{c_1}f(u_1) - \tfrac{1}{c_2}f(v_2)}_{\leq 0}$$

Since $v_1 \in V_0$, $f(v_1) > 0$ by proposition 2. Hence $c_1 = c_2$ and $f(u_1) = f(v_2)$. Using this result we analogously derive from $\Delta(G)f(u_1) = \Delta(G')f(u_1)$, $f(v_1) = f(u_2)$. Thus the proposition follows.□

Now we take an arbitrary d-regular tree with boundary G. Then we can rearrange its edges without increasing the lowest Dirichlet eigenvalue such that the resulting graph is similar to a ball.

Lemma 6:

Let $G(V_0 \cup \partial V, E_0 \cup \partial E)$ be a connected d-regular tree with boundary. Let f be a nonnegative eigenfunction to the first Dirichlet eigenvalue $\nu(G)$. We denote a maximum of f by m, i.e. $f(m) \geq f(v)$ for all $v \in V$. Then by rearranging edges we can construct a d-regular tree with boundary $G'(V, E')$ with properties

(1) $\mu_2(G') = \mu_2(G)$.

(2) $\nu(G') \leq \nu(G)$.

(M1) $|dist(m, u_0) - dist(m, v_0)| \leq 1$ *for all boundary vertices* $u_0, v_0 \in \partial V$ *of* G'.

(M2) $f(u) \leq f(v)$ *if* $dist(m, u) > dist(m, v)$, *for all interior vertices* $u, v \in V_0$.

(M3) $f(u_1) \leq f(u_2)$ *if* $f(v_1) < f(v_2)$ *for all edges* $(u_1, v_1), (u_2, v_2) \in E'$.

PROOF: We construct this graph G' by rearranging the edges of G. This rearrangement will be done by moving pairs of edges stepwise. We start at vertex $v_1 = m$, a maximum of f. Let $W_1 = \{v_1\}$ and $G_1(V, E_1) = G(V, E)$.

In the first step we denote a maximum of f in $V \setminus W_1$ by v_2. Let $W_2 = W_1 \cup \{v_2\}$. If v_2 is adjacent to v_1 we have nothing to do. Otherwise there are vertices $u_1, u_2 \notin W_2$ with $(v_1, u_1), (v_2, u_2) \in E_1$ and $(v_1, v_2), (u_1, u_2) \notin E_1$, since G is a tree. Moreover we can choose these vertices so that either u_1 or u_2 is in the geodesic path from v_1 to v_2. (Figure 3 illustrates the situation. Two cases are possible.) We replace the edges (v_1, u_1) and (v_2, u_2) by (v_1, v_2) and (u_1, u_2). If either (v_1, u_1) or (v_2, u_2) is a boundary edge, then let (u_1, u_2) be a boundary edge of same length. We denote the resulting graph by $G_2(V, E_2)$. Since $f(v_1) \geq f(v_2) \geq f(u_1), f(u_2)$ we can apply lemma 5 and hence $\mu_2(G_2) = \mu_2(G_1)$ and $\nu(G_2) \leq \nu(G_1)$.

In the next step, let v_3 denote a maximum of f in $V \setminus W_2$. Analogously to the first step we connect v_1 and v_3 by an edge. We get a d-regular tree with boundary $G_3(V, E_3)$, with $\mu_2(G_3) = \mu_2(G_2)$ and $\nu(G_3) \leq \nu(G_2)$. In this way we arrive at a d-regular tree with boundary $G_{k_1}(V, E_{k_1})$, where W_{k_1} contains v_1 and all its adjacent vertices. Furthermore for each vertex $v \in W_{k_1}$ and every $u \notin W_{k_1}$, $f(v) \geq f(u)$.

Next we do the same with v_2, i.e. we connect v_2 and v_{k_1+1}, where v_{k_1+1} is a

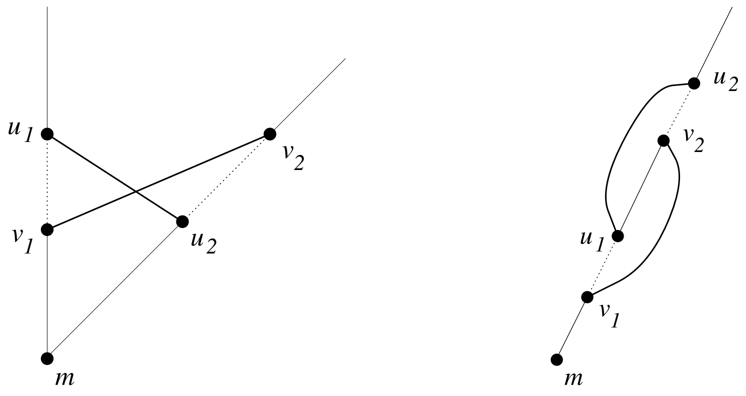

FIGURE 3

maximum of f in $V \setminus W_{k_1}$. Then we connect v_2 with v_{k_1+2}, and so on, until all vertices, that are adjacent to v_2, are in a W_k. Then we continue with v_3, v_4 and all the other vertices adjacent to v_1. We arrive at a graph G_{k_2}, where $\text{dist}(m, v) \leq 2$ if and only if $v \in W_{k_2}$ and where for each vertex $v \in W_{k_2}$ and each $u \notin W_{k_2}$, $f(v) \geq f(u)$.

In the same way we continue until only boundary vertices remain in $V \setminus W_{k_i}$. At last we exchange boundary edges until $f(v) \geq f(u)$ is satisfied whenever boundary edge (v, v_0) is longer than boundary edge (u, u_0) and until (M1) holds. Again we can apply lemma 5 (Now u_1 and v_2 are boundary vertices).

We finish with a d-regular tree with boundary $G'(V, E')$. (1) and (2) holds for each single step by lemma A1. Properties (M1), (M2) and (M3) are satisfied by construction, as claimed.□

We also can decrease the lowest Dirichlet eigenvalue $\nu(G)$ when we change the length of the boundary edges in such a way, that the normal derivative of the eigenfunction to $\nu(G)$ at all boundary edges becomes the same (except at boundary edges of length 1). The normal derivative of f at a boundary edge $(v, v_0) \in \partial E$ of length c_e is given by $\frac{f(v)}{c_e}$.

Lemma 7:
Let $G(V, E_0 \cup \partial E)$ be a connected d-regular tree with boundary and f a nonnegative eigenfunction to the first Dirichlet eigenvalue $\nu(G)$. By changing the length of boundary edges we can construct a d-regular tree with boundary $G'(V, E_0 \cup \partial E')$ which has the properties

(1) $\mu_2(G') = \mu_2(G)$.

(2) $\nu(G') \leq \nu(G)$.

(M4) The normal derivative of f at all boundary edges of length $c'_e < 1$ is the same. It is less than or equal to the normal derivative at each boundary edge of length $c'_e = 1$. Moreover all boundary edges at the same interior vertex have the same length.

(3) Equality in (2) holds if and only if G and f already fulfill property (M4).

PROOF: The normal derivative of f at the boundary edge $e_j = (v_j, u_j) \in \partial E$ of length $c_j = c_{e_j}$ is $\frac{f(v_j)}{c_j}$ ($v_j \in V_0$). Now take n boundary edges of G. The "average" normal derivative is given by

$$\frac{\sum_{j=1}^n f(v_j)}{\sum_{j=1}^n c_j}$$

We replace each of these n edges e_j by edges $\bar{e}_j$ of length $\bar{c}_j$, where each $\bar{c}_j$ satisfies

$$\frac{f(v_j)}{\bar{c}_j} = \frac{\sum_{i=1}^n f(v_i)}{\sum_{i=1}^n c_i} \qquad \Leftrightarrow \qquad \bar{c}_j = f(v_j)\,\frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n f(v_i)} \tag{7.1}$$

Then we have

$$\sum_{j=1}^n \bar{c}_j = \sum_{j=1}^n f(v_j)\,\frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n f(v_i)} = \sum_{i=1}^n c_i$$

i.e. $\mu_2(\bar{G}) = \mu_2(G)$.

Next notice that by (7.1)

$$\sum_{j=1}^n \frac{f(v_j)^2}{\bar{c}_j} = \sum_{j=1}^n f(v_j)\,\frac{\sum_{i=1}^n f(v_i)}{\sum_{i=1}^n c_i} = \frac{\left(\sum_{i=1}^n f(v_i)\right)^2}{\sum_{i=1}^n c_i} \leq \sum_{i=1}^n \frac{f(v_i)^2}{c_i} \tag{7.2}$$

The last inequality follows from inequality 65 in [8], where equality holds if and only if $\frac{f(v_i)}{c_i}$ does not depend on i. Hence $\langle \Delta(\bar{G})f, f\rangle \leq \langle \Delta(G)f, f\rangle$.

It may happen, that $\bar{c}_j > 1$ for a j. Thus $\bar{G}$ is not a d-regular tree. For that reason we replace the edges e_j by edges $e_j(\varepsilon)$ of length $c_j(\varepsilon) = (1-\varepsilon)c_j + \varepsilon\,\bar{c}_j$, where $\varepsilon \in [0,1]$. Denote the resulting graph by $G(\varepsilon)$. Then again $\mu_2(G(\varepsilon)) = \mu_2(G)$. Furthermore inequality

$$\sum_{j=1}^n \frac{f(v_j)^2}{c_j(\varepsilon)} \leq \sum_{j=1}^n \frac{f(v_j)^2}{c_j} \tag{7.3}$$

holds for all $\varepsilon \in [0,1]$, since the left hand side of (7.3) is convex in ε and (7.3) is valid for $\varepsilon = 0$ and $\varepsilon = 1$. Hence $\langle \Delta(G(\varepsilon))f, f\rangle \leq \langle \Delta(G)f, f\rangle$ and thus $\nu(G(\varepsilon)) \leq \nu(G)$. If ε is sufficiently small, then $c_j(\varepsilon) \leq 1$ for all j, i.e. $G(\varepsilon)$ is a d-regular tree.

Now take all boundary edges of G. Construct a graph $G_1(\varepsilon_1)$ as described above with $\varepsilon_1 \in [0,1]$ as great as possible. Then we find $\varepsilon_1 = 1$ or at least one of the boundary edges $c_j(\varepsilon_1)$ has length 1. In the latter case take all boundary edges of $G_1(\varepsilon_1)$ of length less than 1 and construct a graph $G_2(\varepsilon_2)$. Continue until the first time $\varepsilon_k = 1$ occurs.

Let $G' = G_k(\varepsilon_k)$. Then G' is a d-regular tree with boundary which satisfies properties (1), (2) and (M4) by construction. Equality in (7.2) holds if and only if the normal derivative $\frac{f(v_i)}{c_i}$ does not depend on i. Thus (3) follows.□

Yet we have shown methods for decreasing the first Dirichlet eigenvalue $\nu(G)$. Now we prove that on the other side all trees with the Faber-Krahn-property satisfy (M1)–(M4).

Lemma 8:
Let $G(V_0 \cup \partial V, E_0 \cup \partial E)$ be a tree with boundary and f a nonnegative eigenfunction to $\nu(G)$. If G has the Faber-Krahn-property then G and f satisfy properties (M1)-(M4).

PROOF: Property (M4) holds by lemma 7.

If (M3) does not hold, then there exist two edges $(v_1, u_1), (u_2, v_2) \in E$ where $f(u_1) < f(v_2)$ and $f(v_1) > f(u_2)$. Replacing these edges by $(v_1, v_2), (u_1, u_2)$ we get a graph G' with $\nu(G') < \nu(G)$ by lemma 5, a contradiction.

Suppose (M2) does not hold. By applying the rearrangement steps of lemma 6 we get a sequence of d-regular trees G_i with boundary. All these trees have the Faber-Krahn-property and hence $\nu(G_i) = \nu(G_{i+1})$ for each step. Moreover f is an eigenfunction to the first Dirichlet eigenvalue for all i, since f is simple (proposition 2). By lemma 6 there is a k such that $(M2)$ is satisfied for G_{k+1} but not for G_k. In the rearrangement step we then replace the edges $(v_1, u_1), (v_2, u_2) \in E_k$ by $(v_1, v_2), (u_1, u_2)$, where $f(u_1) > f(v_2)$, $\mathrm{dist}(m, u_1) > \mathrm{dist}(m, v_2)$ in G_k and $\mathrm{dist}(m, u_1) \leq \mathrm{dist}(m, v_2)$ in G_{k+1}. Thus $\nu(G_{k+1}) < \nu(G_k)$ by lemma 5, a contradiction.

Now suppose (M1) fails. Again can construct sequence of trees G_i as described above. For every G_i (M2) holds, since every tree has the Faber-Krahn-property.

Moreover there is a k such that $(M1)$ is satisfied for G_{k+1} but not for G_k. For all boundary edges $e_1 = (v_1, w_1), e_2 = (v_2, w_2) \in \partial E_i$, $w_i \in \partial V$, with $\mathrm{dist}(m, v_1) < \mathrm{dist}(m, w_1) \leq \mathrm{dist}(m, w_2) < \mathrm{dist}(m, v_2)$ we have by (M2) and (M4) $c_{e_1} \geq c_{e_2}$. Thus in G_k edges $e_1 = (v_1, u_1) \in \partial E$ and $e_2 = (u_2, v_2) \in E_0$ with $\mathrm{dist}(m, v_1) < \mathrm{dist}(m, u_1) \leq \mathrm{dist}(m, u_2) < \mathrm{dist}(m, u_2) + 1 = \mathrm{dist}(m, v_2)$ exist, since (M1) is not satisfied (see figure 4).

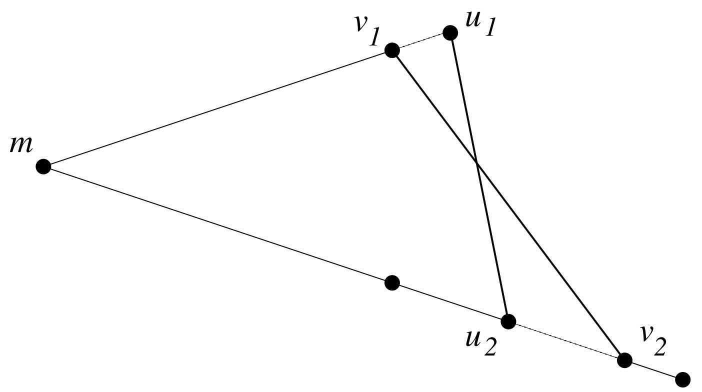

FIGURE 4

For the rearrangement step from G_k to G_{k+1} we have to replace these edges $(v_1, u_1), (v_2, u_2)$ by the edges (v_1, v_2) and (u_1, u_2). Moreover $f(v_1) \geq f(v_2) \geq f(u_2) \geq f(u_1) = 0$ (Otherwise we had not replaced these edges, see proof of lemma 6). Thus by lemma 5, $f(v_1) = f(v_2) = f(u_2) = f(u_1) = 0$, a contradiction to proposition 2.□

PROOF OF THEOREM 1: Immediately from proposition 4 and lemma 8.□

For the case $\nu(G) > 1$ we are able to decrease the Rayleigh quotient again by making long boundary edges longer and short boundary edges shorter.

For this purpose we need some information about $\nu(G)$.

Proposition 9: (see [5])
Let G be a d-regular tree with boundary. Then

$$\nu(G) > d - 2\sqrt{d-1}$$

Lemma 10:
If G consists of exactly two interior vertices and if all $2(d-1)$ boundary edges of G have length 1, then $\nu(G) = d-1$.

PROOF: $\nu(G)$ is the smallest eigenvalue of $\begin{pmatrix} d & -1 \\ -1 & d \end{pmatrix}$.□

Lemma 11:
Let $G(V_0 \cup \partial V, E_0 \cup \partial E)$ be a connected d-regular tree with boundary, $d \geq 5$ and f a nonnegative function on G that satisfies the properties (M1)–(M4). If there exist two boundary edges of length less than 1 which have no common vertices, then we can construct a d-regular tree with boundary $G'(V_0 \cup \partial V, E_0 \cup \partial E')$ which satisfies

(1) $\mu_2(G') = \mu_2(G)$

(2) $\nu(G') < \nu(G)$

PROOF: Assume there exist two such edges. Then by (M4) there are n_1 boundary edges (v_1, u_{1i}) of length $c_1 < 1$ and n_2 boundary edges (v_2, u_{2i}) of length $c_2 < 1$ where $v_1 \neq v_2 \in V_0$. Let $0 < c_1 \leq c_2 < 1$. Without loss assume $f(v_1) = c_1$ and $f(v_2) = c_2$. Let $(v_1, w_1), (v_2, w_2) \in E_0$ be interior edges. By property (M3) $f(w_1) \leq f(w_2)$. (If $f(v_1) = f(v_2)$ and $f(w_1) > f(w_2)$ we change the rôle of v_1 and v_2.) We always can choose v_1, v_2, w_1 and w_2 so that one of the following holds ($n_i = d-1$ means all but one edge at v_i are boundary edges):

(a) $n_1 = n_2 = d-1$ and $(v_1, v_2) \notin E$

(b) $n_1 = d-1$, $n_2 = d-2$ and $w_1 = v_2$ and $w_2 \neq v_1$

(c) $n_1 = n_2 = d-1$ and $|V_0| = 2$

To prove our result, we construct a new graph for the cases (a) and (c), and show that (b) can be avoided.

For case (a) we define for $\varepsilon \in (0, c_1) \cap (0, 1-c_2]$ a function f_ε by $f_\varepsilon(v_1) = f(v_1) - \varepsilon$, $f_\varepsilon(v_2) = f(v_2) + \varepsilon$ and $f_\varepsilon(v) = f(v)$ otherwise. We replace edges (v_1, u_{1i}) and (v_2, u_{2i}) by $(v_1, u_{1i}(\varepsilon))$ and $(v_2, u_{2i}(\varepsilon))$ of lengths $c_1(\varepsilon) = c_1 - \varepsilon$ and $c_2(\varepsilon) = c_2 + \varepsilon$. Denote the resulting graph by $G(\varepsilon)$. Obviously $G(\varepsilon)$ is a d-regular graph with boundary and (1) holds.

Notice that $0 < f(v_1) = c_1 \leq f(v_2) = c_2$ and $0 < f(w_1) \leq f(w_2)$. Then we find

$$\begin{aligned}\langle f_\varepsilon, f_\varepsilon\rangle &= \sum_{v\neq v_1,v_2} f^2(v) + (f(v_1)-\varepsilon)^2 + (f(v_2)+\varepsilon)^2 \\ &= \langle f, f\rangle + 2\,\varepsilon(f(v_2)-f(v_1)) + \varepsilon^2.\end{aligned}$$

and

$$\begin{aligned}\langle \Delta(G(\varepsilon))f_\varepsilon, f_\varepsilon\rangle &= \sum_{e=(u,v)\in E} \frac{1}{c_e}(f_\varepsilon(u)-f_\varepsilon(v))^2 \\ &= \sum_{\substack{(u,v)\in E\\ u,v\notin\{v_1,v_2\}}} \frac{1}{c_e}(f(v)-f(u))^2 \\ &\quad + (f(w_1)-(f(v_1)-\varepsilon))^2 + (f(w_2)-(f(v_2)+\varepsilon))^2 \\ &\quad + \underbrace{\sum_{i=1}^{d-1} \frac{1}{c_1-\varepsilon}(f(v_1)-\varepsilon)^2}_{=\sum \frac{f(v_1)^2}{c_1}-(d-1)\varepsilon} + \underbrace{\sum_{i=1}^{d-1} \frac{1}{c_2+\varepsilon}(f(v_2)+\varepsilon)^2}_{=\sum \frac{f(v_2)^2}{c_2}+(d-1)\varepsilon} \\ &= \langle \Delta(G)f, f\rangle + 2\,\varepsilon(f(v_2)-f(v_1)) + 2\,\varepsilon(f(w_1)-f(w_2)) + 2\,\varepsilon^2 \\ &\leq \langle \Delta(G)f, f\rangle + 2\,\varepsilon(f(v_2)-f(v_1)) + 2\,\varepsilon^2\end{aligned}$$

To verify (2) we have to show that

$$\frac{\langle \Delta(G)f, f\rangle + 2\,\varepsilon(f(v_2)-f(v_1)) + 2\,\varepsilon^2}{\langle f, f\rangle + 2\,\varepsilon(f(v_2)-f(v_1)) + 2\,\varepsilon^2} < \frac{\langle \Delta(G)f, f\rangle}{\langle f, f\rangle}$$

Using the fact that $\nu(G) \leq \frac{\langle \Delta(G)f,f\rangle}{\langle f,f\rangle}$ and that for any positive numbers $x, y, a, b > 0$, $\dfrac{x+a}{y+b} < \dfrac{x}{y} \Leftrightarrow \dfrac{a}{b} < \dfrac{x}{y}$, it remains to show that

$$\frac{2\,\varepsilon(f(v_2)-f(v_1)) + 2\,\varepsilon^2}{2\,\varepsilon(f(v_2)-f(v_1)) + 2\,\varepsilon^2} = 1 < \nu(G)$$

But this immediately follows from proposition 9 for $d \geq 5$.

If $|V_0| = 2$ (case(c)), then we have $(f(v_i) = c_i)$

$$\begin{aligned}\langle \Delta(G(\varepsilon))f_\varepsilon, f_\varepsilon\rangle =& \frac{d-1}{c_1-\varepsilon}(f(v_1)-\varepsilon)^2 + \frac{d-1}{c_2+\varepsilon}(f(v_2)+\varepsilon)^2 + ((f(v_1)-\varepsilon)-(f(v_2)+\varepsilon))^2 \\ =& \langle \Delta(G)f, f\rangle + 4\varepsilon(f(v_2)-f(v_1)) + 4\varepsilon^2\end{aligned}$$

Again it remains to show that

$$\frac{4\varepsilon(f(v_2)-f(v_1))+4\varepsilon^2}{2\varepsilon(f(v_2)-f(v_1))+2\varepsilon^2} = 2 < \nu(G)$$

Since $|V_0| = 2$, G is contained in a graph G' with exactly two interior vertices where all boundary edges have length 1. Hence $\nu(G) > \nu(G') = d-1$ by lemma 10. Thus (2) holds.

Now let $w_1 = v_2$ and $w_2 \neq v_1$ (case(b)). Since f is an eigenfunction we have

$$((d-1)\tfrac{1}{c_1}f(v_1) + f(v_1)) - f(w_1) = \nu(G)f(v_1)$$

Notice that $f(v_1) = c_1 > 0$ and $f(w_1) = f(v_2) = c_2 < 1$

Since $f(v_1) = c_1 < 1$ and $f(v_2) = c_2 < 1$, we arrive at

$$(d-1) = (\nu(G)-1)c_1 + c_2 < \nu(G)$$

Then G cannot contain a graph G' with two interior vertices where all boundary edges have length 1 by proposition 2 and lemma 10. But then we either have situation (a) or situation (c).

This finishes the proof.□

PROOF OF THEOREM 3: Immediately from lemmata 8 and 11.□

For the proof of our last theorem we have to calculate the lowest Dirichlet eigenvalue for balls with a given radius.

Lemma 12:
Let $G(V_0 \cup \partial V, E_0 \cup \partial E)$ be a ball with radius $\varrho = k_\varrho - 1 + c$ and center $p \in V$. Then $\nu(G)$ is the lowest root of

$$(d-\nu)f_{k_\varrho} - d\, f_{k_\varrho - 1} = 0, \textit{ if } \varrho > 1,$$

where $f_1 = c$, $f_2 = (d-1) + (1-\nu)c$ and $f_i = (d-\nu)f_{i-1} - (d-1)f_{i-2}$ for all $i \geq 3$.
If $\varrho \leq 1$, then $\nu(G) = \frac{d}{c}$.

PROOF: Since $\nu(G)$ is simple, $f(v)$ only depends on $\operatorname{dist}(p,v)$. Let $v_k \in V$ denote a vertex with $\operatorname{dist}(v_k, \partial V) = (k-1)+c$. Without loss we can assume $f(v_1) = c = f_1$. Since f is an eigenfunction to $\nu(G)$ we then find $f(v_2) = f_2 = (d-1) + (1-\nu)c$, $f(v_3) = (d-\nu)f_2 - (d-1)f_1$, and so on (see [5], pp.501–502). Thus $f(p) = f_{k_\varrho}$ and $f(v) = f_{k_\varrho - 1}$ for all vertices $v \in V_0$ adjacent to p. Hence the result follows.□

PROOF OF THEOREM 2: The statement is trivial if $d = 2$. If $|V_0| = 1$ or $d \geq 5$, then it holds by lemma 7 and theorem 3, respectively.

Now assume $d = 3$ or $d = 4$. Let G be a ball centered a vertex m and let f be a nonnegative eigenfunction to $\nu(G)$. Then $f(v)$ only depends on $\operatorname{dist}(m,v)$. Let the length of all boundary edges be $c \in (0,1)$, i.e. the length of the boundary length is less than 1. Without loss we assume that $f(v) = c$ for all vertices adjacent to boundary vertices. Now take two branches B_1 and B_2 rooted at m. Define for a sufficiently small $\varepsilon > 0$ a function f_ε by $f_\varepsilon(v_1) = f(v_1) + \varepsilon$ for all $v_1 \in B_1$, $f_\varepsilon(v_2) = f(v_2) + \varepsilon$ for all $v_2 \in B_2$, $f_\varepsilon(m) = f(m)$ and $f_\varepsilon(v) = f(v)$ otherwise. Replace all boundary edges in B_1 by boundary edges of length $c(\varepsilon) = c + \varepsilon$ and all boundary edges in B_2 by boundary edges of length $c(\varepsilon) = c - \varepsilon$. Denote the resulting graph by $G(\varepsilon)$. For sufficiently small ε, $G(\varepsilon)$ is a d-regular tree with boundary. Then analogously to the proof of lemma 11 we find

$$\langle f_\varepsilon, f_\varepsilon \rangle = \langle f, f \rangle + |V_B| \varepsilon^2$$

where $|V_B|$ is the number of interior vertices (except root m) in both branches. Notice that we can map the vertices $v_1 \in B_1$ to vertices $v_2 \in B_2$ one-to-one, so that $f(v_1) = f(v_2)$. Similarly

$$\langle \Delta(G(\varepsilon)) f_\varepsilon, f_\varepsilon \rangle = \langle \Delta(G) f, f \rangle + 2\varepsilon^2$$

Hence it remains to show that

$$\frac{\langle \Delta(G(\varepsilon)) f_\varepsilon, f_\varepsilon \rangle}{\langle f_\varepsilon, f_\varepsilon \rangle} = \frac{\langle \Delta(G) f, f \rangle + 2\varepsilon^2}{\langle f, f \rangle + |V_B| \varepsilon^2} < \nu(G)$$

or equivalently

$$\frac{2}{|V_B|} < \nu(G) \qquad \Leftrightarrow \qquad \nu(G) \cdot |V_B| > 2$$

If G is not contained in a ball of radius 3, then $|V_B| \geq 2d^2 - 2d + 2$. Thus by proposition 9 $\nu(G) \cdot |V_B| > (d - 2\sqrt{d-1}) \cdot (2d^2 - 2d + 2) > 2$ for $d = 3$ and $d = 4$.

If G is contained in a ball K of radius 2, then $|V_B| = 2$ and by proposition 2 and lemma 12 $\nu(G) > \nu(K) = d - \sqrt{d}$. If G is contained in a ball K of radius 3 we find $|V_B| = 2d$ and $\nu(G) > \nu(K) = d - \sqrt{2d-1}$. In both cases we find for $d = 3$ and $d = 4$, $\nu(G) \cdot |V_B| > 2$, as claimed.□

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