On the Number of Nodal Domains of Spherical Harmonics

by

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Abstract

A sharp upper bound for the number of nodal domains of spherical harmonics for the first six eigenvalues is deduced.

1. Introduction

It is a well-known fact that the n-th eigenfunction of the Sturm-Liouville eigenvalue equation $-u'' + V(x)u(x) = \lambda u(x)$, $x \in (0,1)$, $u(0) = u(1) = 0$, has exactly $n-1$ nodes (i.e. non-degenerate zeros), see e.g. [CH].

For the corresponding equation in higher dimension it is much more complicated to obtain general statements on the zero sets of eigenfunctions. We want to illustrate the problems in the case of spherical harmonics. (This is Arnold's problem 1 in [AR2].)

Consider the eigenvalue problem

$$-\Delta u = \lambda u \quad \text{on } S^2 \tag{1}$$

The eigenfunctions are the spherical harmonics with eigenvalues $\lambda = \ell(\ell+1)$, $\ell \in \mathbf{N}_0$. Let $\mu(u)$ denote the number of *nodal domains* of u, i.e. the connected components of $S^2 \setminus \mathcal{N}(u)$, where $\mathcal{N}(u)$ is the *nodal set* of the eigenfunction u of (1), i.e. $\mathcal{N}(u) = \{x \in S^2 \colon u(x) = 0\} = u^{-1}(0)$. In [CS] one finds the *Courant nodal domain theorem*, which states that the number of nodal domains of the n-th eigenfunction does not exceed n (cf. also [CH], §VI.6). By applying this theorem to spherical harmonics we obtain $\mu(u_\ell) \leq \ell^2 + 1$, where u_ℓ denotes a spherical harmonic of degree ℓ, i.e. to the eigenvalue $\ell(\ell+1)$. Using inversionsymmetry of these functions it we can improve this result to

$$\mu(u_\ell) \leq \ell(\ell-1) + 2 \tag{2}$$

Pleijel ([PL], cf. also [CI]) has proved that

$$\limsup_{\ell \to \infty} \frac{\mu(u_\ell)}{\ell(\ell-1)} \leq \frac{4}{j_0^2} < 0.69 \tag{3}$$

where j_0 denotes the smallest zero of the 0-th Bessel function. Consequently equality occurs in (2) for a finite number of values ℓ only. Attempts have been made to improve (2). One approach is based on the Faber-Krahn theorem (see e.g. [CI]).

Another method uses the fact, that the *spherical harmonics* of are the restrictions of the *harmonic homogeneous polynomials* U of degree ℓ in $\mathbf{R}^3$ to the 2-sphere, i.e. of homogeneous polynomials that fulfill $\Delta U = 0$. Thus one can apply results about the zero sets of homogeneous polynomials — called *real projective plane algebraic curves* — and arrives at ([KA], see next chapter)

$$\mu(u_\ell) \le \begin{cases} \ell(\ell-2)+5 & \text{if } \ell \text{ is odd} \\ \\ \ell(\ell-2)+4 & \text{if } \ell \text{ is even} \end{cases} \tag{4}$$

These estimates show, that (2) is not sharp for $\ell \ge 4$. Numerical experiments lead to the conjecture, that the maximum number of nodal domains for a given eigenvalue occur in the case of the functions

$$Y_\ell^m(\theta,\varphi) = c_{\ell m} \begin{cases} P_\ell^m(\cos\theta)\cdot\sin m\varphi & \text{if } 1 \le m \le \ell \\ P_\ell(\cos\theta) & \text{if } m = 0 \\ P_\ell^{|m|}(\cos\theta)\cdot\cos m\varphi & \text{if } -\ell \le m \le -1 \end{cases}$$

where P_ℓ^m denotes the associated Legendre polynomials and $c_{\ell m}$ are normalizing factors. θ and φ denote the spherical coordinates. As can easily be seen, the nodal sets of these spherical harmonics consist of m meridians (great circles through the poles) and $(\ell - m)$ latitude circles (circles with the pole as its center). Thus we have

Conjecture:

$$\max_{u\in E_\ell} \mu(u) = \begin{cases} \frac{1}{2}(\ell+1)^2 & \text{if } \ell \text{ is odd} \\ \\ \frac{1}{2}\ell(\ell+2) & \text{if } \ell \text{ is even} \end{cases} \tag{5}$$

where E_ℓ denotes the eigenspace to the eigenvalue $\ell(\ell+1)$.

In this paper we do not want to deduce a global estimate for the number of nodal domains. We make a detailed investigation of the case of small ℓ and show that the Conjecture holds then. Furthermore we get a glimpse of the problem that arise in applying algebraic geometry to nodal sets.

Theorem: The Conjecture holds for $\ell \le 6$.

Summarizing the estimates we have

method	$\mu(u_2)$	$\mu(u_3)$	$\mu(u_4)$	$\mu(u_5)$	$\mu(u_6)$	$\mu(u_7)$
Courant	4	8	14	22	32	44
Faber-Krahn	4	8	14	21	29	39
Karpushkin	4	8	12	20	28	40
sharp bound	4	8	12	18	24	32(?)

2. Real Algebraic Curves

In the classical model of the real projective plane $\mathbf{R}\mathrm{P}^2$ all points in $\mathbf{R}^3$ which lie on a straight line through the origin are identified. A real projective plane algebraic curve p is the zero set of a homogeneous polynomial $P(x_0 : x_1 : x_2) = \sum_{i+j+k=n} a_{ijk}\, x_0^i x_1^j x_2^k$, where $a_{ijk} \in \mathbf{R}$. The degree of the curve p is defined as the degree n of the polynomial P. We always will use **small letters** (e.g. p) to denote algebraic curves and the corresponding **capital letters** (e.g. P) to denote the homogeneous polynomials of their equations. If we say *components* of p we always mean the topological components of $\mathcal{N}(p) = P^{-1}(0) \setminus \{0\}$. If we say p decomposes or reduces into several *factors*, then we mean that the polynomial P decomposes into factors of P and each factor of p is the curve determined by a factor of P. We also will use the symbol p for the *"spherical polynomial"* $P|_{S^2}$, if there is no risk of confusion.

Now we can use results about real plane algebraic curves and apply them to our problem. But since our nodal sets are curves on the sphere (*"spherical curves"*), not in the projective plane, we have to keep in mind, that

(•) each point in $\mathbf{R}\mathrm{P}^2$ corresponds to a pair of antipodal points on S^2.

(•) there is at most one component of the spherical curve, which remains unchanged under inversion. We call it the *invariant component* of the spherical curve. For all the other components, there exists an antipodal component and this pair of components is counted as two components of $\mathcal{N}(p)$ but as one component of the corresponding projective curve.

(•) there always is an invariant component, if p is of odd degree.

In analogy to the projective plane we call the zero set of linear functions on the 2-sphere (i.e. great circles) *"straight lines"*.

First we introduce a few indices and recall the well-known results being used.

Definition:

Let p be any spherical polynomial. Then we denote

$\mu(p) :=$ the number of nodal domains, i.e. the number of components of the complement of $\mathcal{N}(p)$.

$\varrho_p(x) :=$ the multiplicity of p at x (also called the order of vanishing of p at x).

$\nu_p(x) := \varrho_p(x) - 1$

$\nu(p) := \sum \nu_p(x)$, where the sum is taken over all singular points of p.

$\zeta(p) :=$ number of components of $\mathcal{N}(p)$

Lemma 1: (BEZOUT'S THEOREM)
If two real projective plane algebraic curves p and q of degree m and n have no common factors, then

$$\sum \varrho_p(x) \cdot \varrho_q(x) \leq mn$$

where the sum is taken over all common points of p and q.

Lemma 2: (HARNACK'S THEOREM)
Let p be an irreducible real projective plane algebraic curve. Then the number of components of p does not exceed $g+1$, where g is the genus of the curve p.

Lemma 3: (NOETHER'S THEOREM ABOUT THE GENUS OF A CURVE)
Let p be an irreducible real projective plane algebraic curve of degree n. Then its *genus* is given by

$$g = \frac{(n-1)(n-2)}{2} - \sum \frac{\varrho_p(x)\,(\varrho_p(x)-1)}{2}$$

where the sum is taken over all singular points x of the projective curve p.

Lemma 4: (RESULTANT OF TWO POLYNOMIALS)
Let $F_n = A_n + A_{n-1}x_2 + \ldots + A_0 x_2^n$ and $G_m = B_m + B_{m-1}x_2 + \ldots + B_0 x_2^m$, where A_i, B_i are homogeneous polynomials of degree i in x_0, x_1. Then

$$R(x_0,x_1) = \begin{vmatrix} A_n & A_{n-1} & \ldots & \ldots & A_0 & & & \\ & A_n & A_{n-1} & \ldots & \ldots & A_0 & & \\ & & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & & A_n & A_{n-1} & \ldots & \ldots & A_0 \\ B_m & B_{m-1} & \ldots & \ldots & B_0 & & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & & B_m & B_{m-1} & \ldots & \ldots & B_0 \end{vmatrix}$$

is called the *resultant* of F and G with respect to x_2. It has the following properties:

(a) $R(x_0,x_1)$ is either a homogeneous polynomial of degree mn or equal to zero.

(b) $R(c_0,c_1) = 0$ if and only if there is a c_2 so that $(c_0 : c_1 : c_2)$ is a common point of the curves f and g.

(c) $R(x_0,x_1) \equiv 0$ if and only if f and g have a common factor.

A good survey of real projective algebraic curves can be found in the papers of D. A. Gudkov [GU] and O. Ya. Viro [VO]. Results and definitions on (complex) projective algebraic curves can be found in the book of R. Walker [WR] and the wonderful text-book of E. Brieskorn and H. Knörrer [BK]. There proofs for these classical theorems are given (cf. also [HA], [VW], [WI]).

Additionally we introduce an inequality for spherical curves (its proof, a simple application of Euler's theorem for planar graphs, is given later).

Lemma 5: (EULER'S THEOREM FOR SPHERICAL CURVES)
Let p be a spherical polynomial, then

$$\mu(p) - \nu(p) - \zeta(p) \leq 1$$

We can now use these results to deduce an upper bound for the number of nodal domains of arbitrary spherical polynomials. We immediately have

Proposition 1:
Let p be an irreducible spherical polynomial of degree n. Then

$$\mu(p) \leq (n-1)(n-2) + \kappa_n + 1$$

where

$$\kappa_n = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Proof:
Suppose n is odd. Then by Harnack's theorem $\zeta(p) \leq 2g+1$. Since $\varrho_p(x) \geq 2$ for each singular point of p we have

$$\zeta(p) + \nu(p) = 2g + 1 + 2\sum(\varrho_p(x) - 1) \leq 2g + 1 + 2\sum \frac{\varrho_p(x)(\varrho_p(x)-1)}{2}$$

Thus the proposition follows from Noether's and Euler's theorem.

The proof for even n is the same, except then there might not be an invariant component of the spherical curve p.□

Now we use Bezout's theorem to calculate the maximum number of nodal domains of decomposable polynomials.

Lemma 6:

Let p_m, p_n be two spherical curves of degree m and n without a common factor. Then

$$\nu(p_m \cdot p_n) \leq \nu(p_m) + \nu(p_n) + 2mn \tag{6}$$

$$\zeta(p_m \cdot p_n) \leq \zeta(p_m) + \zeta(p_n) \tag{7}$$

Equality holds in (6) if and only if p_m and p_n have exactly $2mn$ common points, such that no one of these is a singular point of p_m or p_n. At least one the inequalities (6) and (7) is strict.

Proof:

Let S_m and S_n be the set of singular points of p_m and p_n, S_c the set of common points of p_m and p_n. Then $S = S_c \cup S_m \cup S_n$ contains all singular points of $p_m \cdot p_n$. Since for any point x, $\varrho_{p_1 \cdot p_2}(x) = \varrho_{p_1}(x) + \varrho_{p_2}(x)$ we have

$$\begin{aligned}
\nu(p_m \cdot p_n) &= \sum_{S_m \setminus S_c} \varrho_{p_m}(x) + \sum_{S_n \setminus S_c} \varrho_{p_n}(x) + \sum_{S_c} (\varrho_{p_m}(x) + \varrho_{p_n}(x)) - |S| \\
&= \underbrace{\sum_{S_m \setminus S_c} (\varrho_{p_m}(x) - 1)}_{\leq \nu(p_m)} + \underbrace{\sum_{S_n \setminus S_c} (\varrho_{p_n}(x) - 1)}_{\leq \nu(p_n)} + \sum_{S_c} (\varrho_{p_m}(x) + \varrho_{p_n}(x) - 1)
\end{aligned}$$

By Bezout's theorem we have

$$\sum_{S_c} (\varrho_{p_m}(x) + \varrho_{p_n}(x) - 1) \leq \sum_{S_c} \varrho_{p_m}(x) \varrho_{p_n}(x) \leq 2mn$$

and the first inequality follows. The condition for equality in (6) are obvious from the above considerations. Inequality (7) is trivial.□

Remark:

If p has any multiple factor, then $\nu(p) = \infty$.

Theorem 1:

Let p be a spherical polynomial of degree ℓ. Then

$$\mu(p) \leq \ell(\ell - 1) + 2$$

Proof:

Because of proposition 1 it remains to show the theorem for decomposable polynomials. Assume $p = p_m \cdot p_n$. Without loss of generality p does not have any multiple factors and thus p_m and p_n does not have a common factor. By induction, Lemma 6 and Proposition 1

$$\begin{aligned}\nu(p) + \zeta(p) &< (\nu(p_m) + \zeta(p_m)) + (\nu(p_{\ell-m}) + \zeta(p_{\ell-m})) + 2m(\ell - m)\\ &\le m(m-1) + 1 + (\ell - m)(\ell - m - 1) + 1 + 2m(\ell - m)\\ &= \ell^2 - \ell + 2\end{aligned}$$

and thus by Euler's theorem $\mu(p) \le \nu(p) + \zeta(p) + 1 \le \ell(\ell - 1) + 2$, as claimed.□

Remark:

It is obvious that equality holds if and only if p consists of ℓ straight lines (linear factors), such that no more than two of them intersect in one point (Lemma 6).

Remark:

It is a curious fact, that this estimate — derived from algebraic results — is identical to Courant's estimate (2), an analytic result for solutions of (Dirichlet or closed) eigenvalue problems. The same phenomenon occurs for eigenfunctions of the harmonic oscillator in two dimensions (Hermite-polynomials). V. I. Arnol'd suggested in [AR1] the following method to proof theorem 1. Homogeneous polynomials of degree n can be expressed as sums of harmonic homogeneous polynomials of degree less than or equal to n. Then one can apply the Courant-Hermann theorem ([CH], §VI.6), a generalization of Courant's theorem. He noticed that this theorem is *false* in general but if it can be proved for the sphere with the standard metric, theorem 1 follows (for details see [GU], §9.2). But this does not enlighten the situation.

Therefore we need more information on nodal lines of spherical harmonics. A very important fact is the simple topology of singular points.

Lemma 7: (BERS' THEOREM)

If k local branches of the nodal lines of a spherical harmonic u intersect in a point x, then the tangents to these branches at x form an equiangular system. Moreover, x is a point of multiplicity k, i.e. $\varrho_u(x) = k$.

We say that a spherical polynomial has the *Bers property*, if it satisfies Bers' theorem. The proof of this theorem can be found in [CS].

Due to this fact, the inequality in Lemma 5 becomes an identity for harmonic homogeneous polynomials.

Lemma 8: (EULER'S THEOREM FOR SPHERICAL HARMONICS)
Let u be a spherical harmonic. Then

$$\mu(u) - \nu(u) - \zeta(u) = 1$$

Proof:
It is well known, that the nodal lines of spherical harmonics are smooth curves. Now if $\mathcal{N}(u)$ is connected, we obtain a cellular decomposition of the sphere, where the 2-cells are the nodal domains of u and the nodal lines form the 1-skeleton of this finite CW complex. We split each of the closed curves in this cellular complex, so that the closure of each 1-cell (i.e. an arc without double points) contains exactly two 0-cells (i.e. the points of the cellular complex). For this we can use Euler's theorem (see e.g. [VJ], p. 73) to obtain

$$\#\text{ 0-cells} - \#\text{ 1-cells} + \#\text{ 2-cells} = \chi(S^2) = 2$$

Obviously $\mu(u) = \#$ 2-cells. From Bers' theorem we know that the curves form an equiangular system with $\varrho_u(x)$ branches when they meet at a point x. So any 0-cell is in the closure of exactly $2\varrho_u(x)$ 1-cells. Since the closure of any 1-cell contains exactly two 0-cells, we conclude that $\sum \varrho_u(x)$, taken over all 0-cells, is the number of 1-cells. Since $\nu_p(x) := \varrho_p(x) - 1$, we obtain $\nu(u) = \sum(\varrho_u(x) - 1) = \#$ 1-cells $-$ $\#$ 0-cells, and the result follows.

If $\mathcal{N}(u)$ is not connected, we need exactly $(\zeta(u) - 1)$ arcs, which join the $\zeta(u)$ components, to obtain a finite CW complex. Considering these supplementary lines, the result follows.□

Proof of Lemma 5:
The zero set of a spherical polynomial consists of distinct points and smooth one dimensional manifolds, which connect (some of) the points. So we again obtain a cellular complex. But in this case each 0-cell is in the closure of *at most* $2\varrho_p(x)$ 1-cells. So if p has no multiple factors, the result follows.□

Karpushkin ([KA]) used Bers' theorem to improve Courant's estimate. Using his results we arrive at (4). This estimate is not sharp for $\ell > 4$.

Our aim is to use Bers' theorem and try to derive some sharp bounds for the number of nodal domains for small degree, in detail for the cases $\ell \leq 6$. We do this by decomposing all spherical harmonics into irreducible factors and distinguish between the different cases.

For each set of irreducible factors we estimate the number of nodal domains. By using differents tricks we are able to prove our theorem. The derived bound is sharp, but it is (probably) not sharp for some cases we deal with.

Lemma 9:
Let p_m and p_n be factors of degree m and n of a spherical harmonic. Then

$$\mu(p_m \cdot p_n) \leq \mu(p_m) + \mu(p_n) + 2mn - \kappa'$$

where

$$\kappa' = \begin{cases} 2 & \text{if both } p_m \text{ and } p_n \text{ have an invariant component} \\ 3 & \text{otherwise} \end{cases}$$

Equality holds only if

p_m and p_n have exactly $2mn$ common points,
$\varrho_{p_m}(x) = \varrho_{p_n}(x) = 1$ at each of these points and
$\zeta(p_m \cdot p_n) = \zeta(p_m) + \zeta(p_n) - 1$ if p_m or p_n has an invariant component
and $\zeta(p_m \cdot p_n) = \zeta(p_m) + \zeta(p_n) - 2$ otherwise.

Proof:
From the Lemmata 6 and 8 (Euler's theorem) we already have

$$\begin{aligned} \mu(p_m \cdot p_n) &= \nu(p_m \cdot p_n) + \zeta(p_m \cdot p_n) + 1 \\ &\leq (\nu(p_m) + \zeta(p_m) + 1) + (\nu(p_n) + \zeta(p_n) + 1) - 2 + 1 + 2mn - (\kappa' - 1) \\ &= \mu(p_m) + \mu(p_n) + 2mn - \kappa' \end{aligned}$$

where κ' is an appropriate integer greater than 1, since at least one of the inequalities in Lemma 6 is strict.
It is obvious, that $\kappa' = 2$ is not possible if $\zeta(p_m \cdot p_n) \leq \zeta(p_m) + \zeta(p_n) - 2$ or if p_m of p_n have no common points. Now assume (without loss) p_m does not have an invariant component and the component C of p_m intersects p_n. Then the antipodal component $\tilde{C}$ intersects p_n also. Thus the total number of components of $p_m \cdot p_n$ cannot exceed $\zeta(p_m) + \zeta(p_n) - 2$ and $\kappa' \geq 3$.
The condition for equality can be deduced from Lemma 6.□

Now we split the spherical harmonics into their irreducible factors. For each of these factors f_n of degree n we have by Proposition 1

$n =$	1	2	3	4	5	6	...
$\mu(f_n) \leq$	2	3	4	9	14	23	...

We now apply Lemma 9 to all the different cases of spherical harmonics of degree $\ell = 6$ and arrive at

Case	*factors*	$\mu(u_6) \leq$
(i)	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$	32
(ii)	$2 \cdot 1 \cdot 1 \cdot 1 \cdot 1$	30
(iii)	$2 \cdot 2 \cdot 1 \cdot 1$	28
(iv)	$2 \cdot 2 \cdot 2$	27
(v)	$3 \cdot 1 \cdot 1 \cdot 1$	28
(vi)	$3 \cdot 2 \cdot 1$	26
(vii)	$3 \cdot 3$	24
(viii)	$4 \cdot 1 \cdot 1$	26
(ix)	$4 \cdot 2$	25
(x)	$5 \cdot 1$	24
(xi)	6	23

(The numbers in the second column separated by dots indicate the degree of the factors of a spherical harmonic.) Obviously there is nothing to do for the cases **(vii)**, **(x)** and **(xi)**. For all the other cases additional work must be done.

3. Linear Factors

First assume we have at least one linear factor. Then there exist some useful symmetry relations.

Lemma 10:

Let L be a linear factor of the harmonic homogeneous polynomial U. Then U is skew-symmetric with respect to l.

Proof:

Without loss we can assume $U(x) = x_0 \cdot A(x)$ and we have to show, that $A(-x_0, x_1, x_2) = A(x_0, x_1, x_2)$. Since U is harmonic,

$$\Delta^k U = x_0 \cdot \Delta^k A(x) + 2k \frac{\partial}{\partial x_0} \Delta^{k-1} A(x) = 0$$

holds for all $k \geq 1$. Thus all powers of x_0 in A must be even.□

Corollary 10.1:

Let u be a spherical harmonic of even degree, which decomposes into at least one linear factor, then $\mu(u)$ is an integer multiple of 4.

Proof:

Again let $U(x) = x_0 \cdot A(x)$. Then $A(x)$ is a homogeneous polynomial of odd degree and therefore $A(x_0, -x_1, -x_2) = A(-x_0, x_1, x_2) = A(x_0, x_1, x_2)$, i.e. $U(x)$ is skew-symmetric with respect to the axis $(1, 0, 0)$, the center of the hemisphere cut out by the nodal line $x_0 = 0$. Thus the result follows.□

Corollary 10.2:

Let u be a spherical harmonic which has at least one linear factor. Denote $k(x)$ the number of linear factors which intersect each other in $x \in S^2$. Then $\mu(u) \equiv 0 \bmod 2k(x)$.

Proof:

Immediately from Lemma 10 and Bers' theorem.□

Another consequence of Lemma 10 is

Lemma 11: (THEOREM ABOUT SPHERICAL HARMONICS AND PLATONIC SOLIDS)
Let u be a spherical harmonic, which completely decomposes into linear factors. Then the nodal set of u is the union of ℓ different straight lines which either satisfy one of the two properties

(a) All lines intersect in two antipodal points, i.e. u possesses one pair of singular point of multiplicity ℓ.
(b) u possesses one pair of singular point of multiplicity $(\ell-1)$ at the pole of the sphere and $2(\ell-1)$ singular points of multiplicity 2 situated at the equator.

or is one of the exceptions

(c) $\ell = 6$ and u possesses 8 singular points of multiplicity 3 and 6 singular points of multiplicity 2.
(d) $\ell = 9$ and u possesses 6 singular points of multiplicity 4, 8 singular points of multiplicity 3 and 12 singular points of multiplicity 2.
(e) $\ell = 15$ and u possesses 12 singular points of multiplicity 5, 20 singular points of multiplicity 3 and 30 singular points of multiplicity 2.

The location of these straight lines is closely related to the regular polyhedra (*"Platonic solids"*) (see Figure 1).

Lemma 12:
The product u_{lin} of all linear factors of a spherical harmonic u is a spherical harmonic.

For the proofs of Lemmata 11 and 12 see [LJ].

Now we can continue to prove our Theorem.

Because of Corollary 10.1 the desired estimate holds for the cases **(vi)** and **(viii)**.

For the case **(v)** either three lines intersect in a point and thus we have $\mu(u) \equiv 0 \bmod 6$ (Corollary 10.2), or $\mu(u) \equiv 0 \bmod 8$ (by applying Lemma 10 to each line). Hence $\mu(u) \leq 24$.

In the same way we have for **(ii)** that either four or three great circles intersect in one point (Lemmata 11 and 12). In the first case $\mu(u) \equiv 0 \bmod 8$, in the second case we have $\mu(u) \equiv 0 \bmod 12$ and the result follows.

The case **(i)** immediately follows from Lemma 11.

This leaves us with the remaining cases **(iii)**, **(iv)** and **(ix)**.

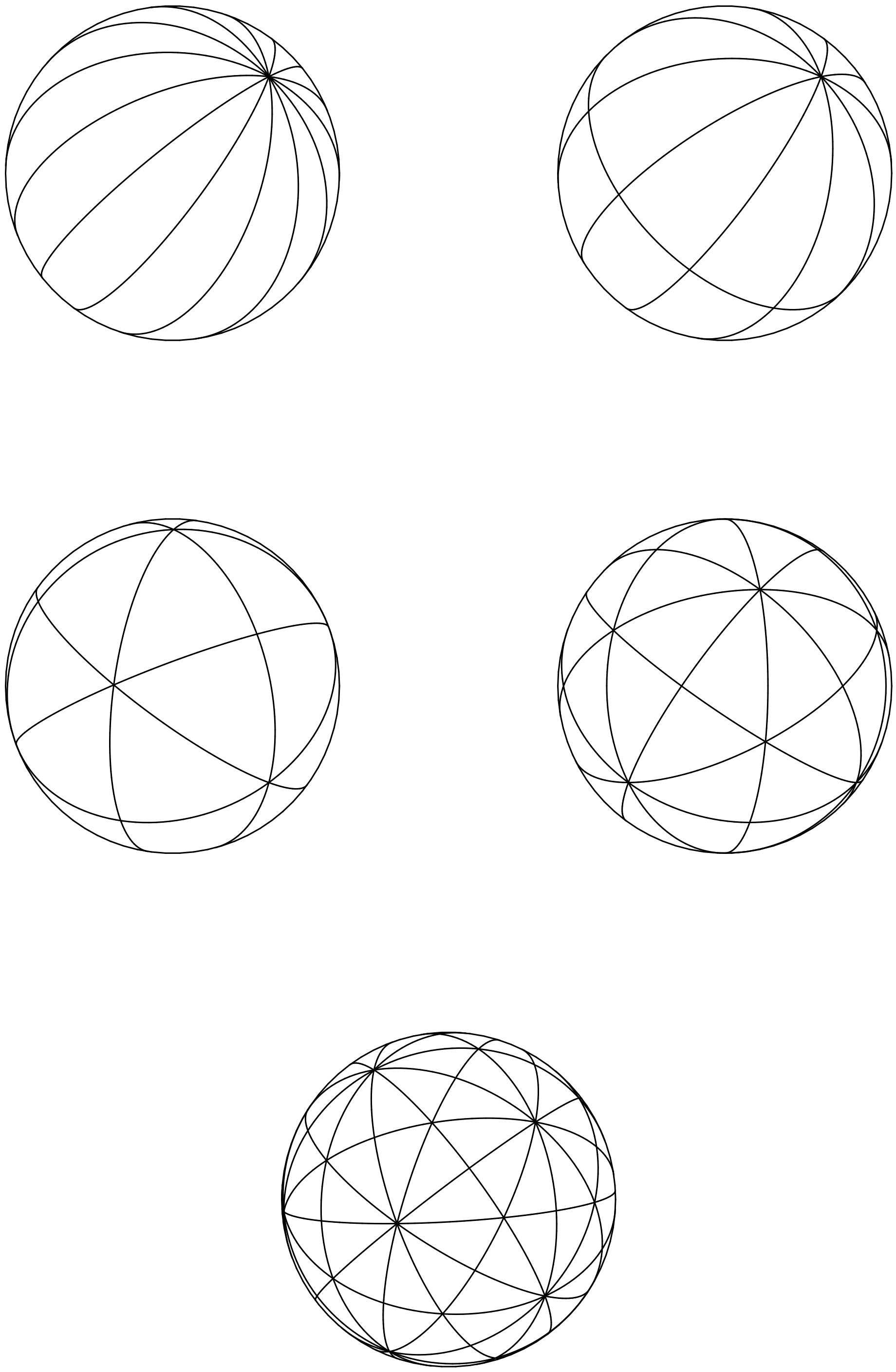

Figure 1: Spherical harmonics of types (a) – (e)

4. Quadratic factors

To describe the possible situations of conics, which satisfy the Bers property we define two distinguished points, namely the *center* and the *saddle* of a conic.

Every homogeneous polynomial P of degree two in $\mathbf{R}^3$ can be written as a quadratic form $P(x) = \langle x, A_P x\rangle$. The three eigenvectors e_i to the eigenvalues λ_i of the symmetric matrix A_P are orthogonal to each other and these are — normalized to one — the stationary points of the function p (at these points the gradient of P is a multiple of the point itself and hence perpendicular to the sphere). Since we are interested in the factors of spherical harmonics, $p^{-1}(0) \neq \emptyset$ and P must change sign at its nodal set. Hence there must be a positive and a negative eigenvalue of A_P and we can assume without loss that $\lambda_1 < 0$, $\lambda_3 \geq \lambda_2 \geq 0$. (The case when the zero set of P is empty is not of any interest for us and hence we can ignore it.) If $\lambda_2 = 0$, then p contains a singular point and hence is reducible, i.e. it consists of a pair of lines (This follows immediately from Harnacks theorem, since then $\zeta(p) + \nu(p) = 3$).

If $\lambda_2 \neq \lambda_3$, then we call e_2 the *saddle point* of the conic p. (From Morse theory we know, that this really is a saddle point of the function p.)
If $\lambda_2 \neq 0$, i.e. p is not irreducible, then e_1 lies in the inside of the "oval" and we call it the *center* of the conic p.

Now we can use these points to describe the location of two conics having the Bers property and four common points.

Proposition 2:
Let p and q be two irreducible conics which have four different pairs of real common points. Suppose p and q are orthogonal to each other, i.e. their tangents at these points are orthogonal. Then the center of p is the saddle of q and the saddle of p must be inside of q (or vice versa). Furthermore neither p nor q is a circle.
If moreover the stationary points of p and q coincide, then the center of one is the saddle of the other and vice versa.

Proof: See Appendix

Corollary 2.1:
If two conics have a common center, they cannot intersect orthogonally.

Proof:

It can easily be seen from symmetry, that the two conics have either four or no common points. By Proposition 2 the two conics cannot intersect each other orthogonally in four points.□

For the intersection of a conic with a straight line we find

Lemma 13:

If a straight line l intersects a conic p orthogonally, then either

(1) l is an axis of the conic p, i.e. a straight line through the vertices of the conic (= through its center and another eigenvalue of A_P); or

(2) p is a circle and l any straight line through its center.

Proof:

Assume a conic $P(x) = \langle x, A_p x\rangle = 0$ with stationary points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ (its center), and a line $L(x) = \langle a, x\rangle = 0$, which intersects p orthogonally, that is $L'(x) = \langle \nabla L, \nabla P(x)\rangle = \langle a, A_p x\rangle = 0$ at the common points of l and p. Hence the lines l and l' must be equal and thus $L' = \nu L$ for some $\nu \in \mathbf{R}$. Hence by comparison of coefficients the result follows.□

For the case **(iii)** we can now deduce from Proposition 2 and Lemma 13: Either both conics intersect both great circles and have a common center; or the two conics have four common pairs of points and thus one conic cannot intersect both straight lines. Thus the estimate follows.

For the case **(iv)** we need a further corollary of Proposition 2.

Corollary 2.2:

If three conics p, q_1 and q_2 intersect each other orthogonally, where p intersects each of the other conics q_i in four pairs of points, then q_1 and q_2 have at most two pairs of common points.

Proof:

Suppose q_1 and q_2 also have four pairs of points in common. Then we know from Proposition 2, that they can not have a common center or a common saddle. Hence (without loss) the center of p is the saddle of q_1 and the saddle of p is the center of q_2. But then the center of q_1 must be the saddle of q_2 by Proposition 2. Since the center and the saddle of a conic must be orthogonal the stationary points of the three conics coincide. Hence by the second part of Proposition 2 the saddle of p must be the center of q_1 and of q_2. Hence q_1 and q_2 have a common center, a contradiction.□

Thus in case **(iv)** more than 24 nodal domains can occure only if each conic intersects each other in four pairs of common points and if there is exactly one point, which is common to all three conics, and if there is no invariant component of u (Lemma 9). But then one component of one conic intersects just one component of the others. The tangents at the common points would create an convex octagon with angle sum less than $3\frac{\pi}{2}+\frac{1\pi}{3}+4\pi < 6\pi$, a contradiction since the angle sum of an octagon on a sphere must be greater than 6π.

Now only the case **(ix)** is left. We first show

Lemma 14:
Let u be a spherical harmonic of degree 6 which contains a latitude circle (which is not the equator). Then $u(\theta\varphi) = P_\ell^m(\cos\theta)(\alpha_1 \sin m\varphi + \alpha_2 \cos m\varphi)$ for an index m and a proper choice of α_i.

Proof:
As can be checked (numerically) two of the Legendre-polynomials P_6^m, $0 \le m \le 6$ have no common zero except 0. Thus the result follows from the linear independence of the functions $\sin m\varphi$ and $\cos m\varphi$.□

Lemma 15:
Let p be any proper conic with non-empty image and y be an arbitrary point on the sphere not on the conic. Then either p is a circle with center y, or there are at most four pairs of points $x_i \in p$, such that the straight line through y and x_i is perpendicular to the tangent on p at x_i, for each i.

Proof:
Let $P(x) = \langle x, A_p x\rangle = 0$ the equation of the conic and $L_{x,y}(x') = 0$ the equation of the straight line through y and an arbitrary point x. Then the points x_i must fulfill $P(x) = 0$ and $Q(x) := \langle \nabla L, \nabla P\rangle = 0$. Since $\nabla P(x) = 2\,A_p x$ and $\nabla L_{xy} = y \times x$ (where $\times$ denotes the cross product), Q is a polynomial of degree two and the result follows from Bezout's theorem.□

5. The last case (ix)

Let $u = u_4^{irr} \cdot u_2^{irr}$.

First we deal with the case, where u_4^{irr} is symmetric with respect to the axes of u_2^{irr}. Under this circumstances we can use Lemma 15 to show that our theorem holds. It turns out that nearly all harmonic products of conics and quartics are symmetric. The few exceptions can be calculated explicitly.

We already know that $\mu(u_4^{irr} \cdot u_2^{irr}) \leq 25$. By Lemma 9 equality can hold only

- if $u_4^{irr} \cdot u_2^{irr}$ has not an invariant component,
- if just one pair of antipodal components of u_4^{irr} intersects with u_2^{irr} — we denote one of these components with $\tilde{C}$,
- if u_4^{irr} and u_2^{irr} have 8 different common points, which are not singular points of u_4^{irr},
- if u_4^{irr} has 9 nodal domains,
  and hence if u_4^{irr} has no singular points of multiplicity 3 or higher (the theorems of Harnack and Noether),
- if u_2^{irr} is not a circle (Lemma 14).

We will show that this is impossible.

(I) First we note, that there must be a component C of u_4^{irr}, which does not intersect the conic u_2^{irr}. Otherwise u_4^{irr} has just one pair of components and 3 pairs of singular points which either are located on one axis of the conic or these are the stationary points of u_2^{irr}. Both cases imply that u_4^{irr} has an invariant component, which contradicts our assumptions.

(II) Now take an arbitrary point a from the inside of this component. Each line through this point cuts this components at least twice and hence by Bezout's theorem it intersects the component $\tilde{C}$ in at most two points.

(III) As can easily be seen, C cannot be inside of $\tilde{C}$. Otherwise there neither are other components besides C and $\tilde{C}$ (and their antipodal sets), nor there are any singular points of u_4^{irr}. Thus u_4^{irr} would not have nine nodal domains.

(IV) Because of (II) and (III) we can split $\tilde{C}$ into two connected arcs, such that both intersect any straight line through the point a (inside of C) at most once. Now take two common points x_1 and x_2 of one arc and the conic, which are "neighboring" on the conic (see Figure 2).

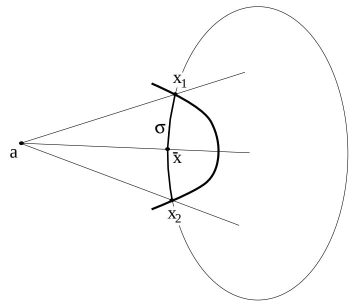

FIGURE 2

(V) The straight lines through a and the common points x_1 and x_2 cannot be orthogonal to the conic at these points. Otherwise they must be inflectional tangents to u_4^{irr}, a contradiction to (II).

(VI) Let σ denote the arc on the conic between x_1 and x_2, and l_y the line through a and an arbitrary point on σ. Then the angle between the tangent of the conic at y and l_y depends continuously on y. Assume each line l_y intersects σ only once. Then the cosines of this angle at x_1 and x_2 have different sign and hence there exists a point $\bar{x} \in \sigma$, where $l_{\bar{x}}$ intersects the conic orthogonally (see Figure 2).

(VII) Because of the symmetric properties all common points, which are neighbored on the conic, are neighbored on u_4^{irr} (i.e. there exist arcs on the curves which only contains these common points).

(VIII) As can easily be seen, there are at most two common points, where the arc σ (defined in (VI)) intersects an straight line l_y more than once.

(IX) There are eight common points of u_4^{irr} and the conic. Because of (VI), (VII) and (VIII) there exist at least 6 points $\bar{x}_i$ on the conic, where the straight lines $l_{\bar{x}_i}$ intersect orthogonally. But this is a contradiction to Lemma 15.

Therefore $u = u_4^{irr} \cdot u_2^{irr}$ cannot have 25 nodal domains if it is symmetric with respect to the axes of u_2^{irr}.

At last we calculate all spherical harmonics of case **(ix)**, which do not have such a symmetry property.

Let $U = Q \cdot P$ be a harmonic homogeneous polynomial of degree ℓ, where $Q = \alpha x_0^2 + \beta x_1^2 + \gamma x_2^2$ is irreducible and $P = \sum_{i+j+k=\ell-2} a_{ijk}\, x_0^i x_1^j x_2^k$. Since U is harmonic, all coefficients of the polynomial ΔU must be zero and thus the coefficients of P and Q satisfy

$$A(\alpha, \beta, \gamma) \cdot \vec{a} = 0 \tag{8}$$

where $A(\alpha, \beta, \gamma)$ is a matrix, depending on α, β and γ only, and $\vec{a}$ contains the coefficients a_{ijk}. At least one of these coefficient must be different from zero (otherwise $U \equiv 0$). Thus we have

$$(\alpha, \beta, \gamma) \in (\det(A(\alpha, \beta, \gamma)))^{-1}(0) =: \Xi^{-1}(0) \tag{9}$$

$$\vec{a} \in \ker(A(\alpha, \beta, \gamma)) \tag{10}$$

We can simplify the search for quadratic factors of harmonic polynomials by using the following method:

We split the space of homogeneous polynomials of degree ℓ into the subspaces of polynomials which are (skew-) symmetric with respect to the axes of the conic q. Any harmonic homogeneous polynomial U can be written as sum of four harmonic polynomials U_{ij},

$$U = U_{00} + U_{01} + U_{10} + U_{11}$$

where $U_{ij}\big((\pm x_0, \pm x_1, x_2)\big) = (\pm 1)^i (\pm 1)^j U_{ij}\big((x_0, x_1, x_2)\big)$. If U has the factor Q, then U_{ij} either contains the factor Q or is zero. We now can derive an algebraic curve (9) for each of these subspaces, which we denote by $\xi_{ij} = \Xi_{ij}^{(-1)}(0)$. Thus we have $\Xi = \Xi_{00} \cdot \Xi_{10} \cdot \Xi_{01} \cdot \Xi_{11}$ (Figure 3 shows the curve ξ for $\ell = 6$).

From now on let $\ell = 6$.

If Q occurs as factor of a harmonic homogeneous polynomial, which has the symmetric property, its coefficients must be a point of the corresponding algebraic curve ξ_{ij}.

If Q occurs as a factor of a non-symmetric harmonic polynomial, its coefficients must be a point of at least two curves ξ_{ij}. Since we only are interested in such polynomials, we have to find all common points of all pairs of these curves. Because of Lemma 14 we only are interested in those points, where $\alpha \neq \beta$ (otherwise q is a circle) and where all coefficients are $\neq 0$ (otherwise q is decomposable). Thus (without loss of generality) we have

$$\alpha, \beta > 0, \alpha \neq \beta \text{ and } \gamma < 0 \tag{11}$$

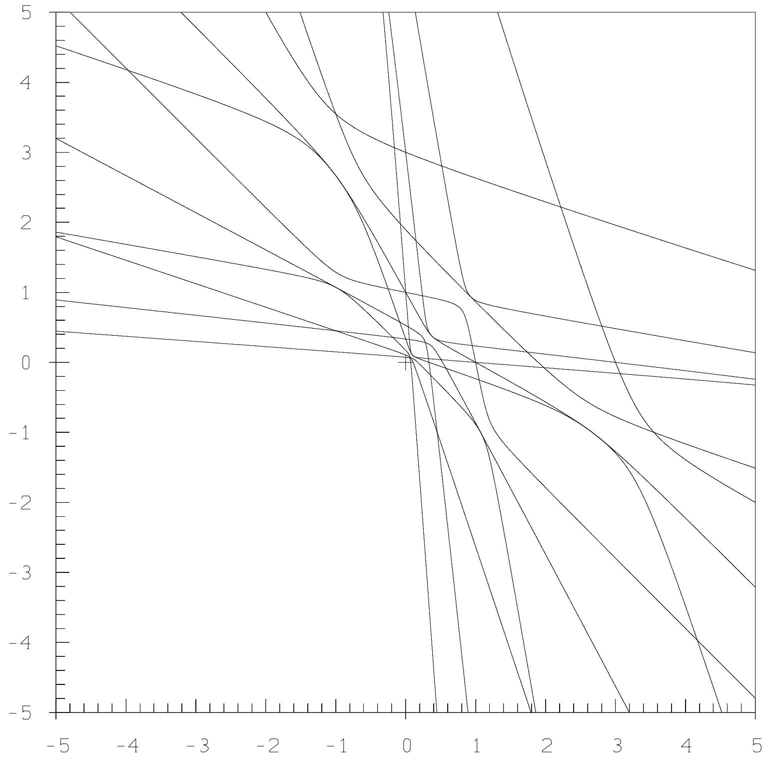

Figure 3: The curve $\xi_{00} \cdot \xi_{10} \cdot \xi_{01} \cdot \xi_{11}$ plotted for $\gamma = -1$, $-5 \leq \alpha, \beta \leq 5$.

Assume $U_{10} \neq 0$. Then

$$U_{10} = x_0 x_2 (\alpha x_0^2 + \beta x_1^2 + \gamma x_2^2)(\alpha' x_0^2 + \beta' x_1^2 + \gamma' x_2^2)$$

We can state equation (8) and obtain for the curve ξ_{10}

$$\Xi_{10} := \begin{vmatrix} 10\alpha + \beta + 3\gamma & \alpha & 3\alpha \\ \beta & \alpha + 2\beta + \gamma & \beta \\ 3\gamma & \gamma & 3\alpha + \beta + 10\gamma \end{vmatrix}$$

$$= \beta^3 + 13(\alpha + \gamma)\beta^2 + 35(\alpha^2 + \gamma^2)\beta + 106\alpha\beta\gamma + 65\alpha\gamma(\alpha + \gamma) + 15(\alpha^3 + \gamma^3)$$

Equally we have

$$\Xi_{11} := \gamma^3 + 13(\alpha + \beta)\gamma^2 + 35(\alpha^2 + \beta^2)\gamma + 106\alpha\beta\gamma + 65\alpha\beta(\alpha + \beta) + 15(\alpha^3 + \beta^3)$$

$$\Xi_{01} := \alpha^3 + 13(\beta+\gamma)\alpha^2 + 35(\beta^2+\gamma^2)\alpha + 106\alpha\beta\gamma + 65\beta\gamma(\beta+\gamma) + 15(\beta^3+\gamma^3)$$

and

$$\begin{aligned}\Xi_{00} :=& 5\gamma^6 + \\ & (110\alpha + 110\beta)\gamma^5 + \\ & (635\alpha^2 + 2070\alpha\beta + 635\beta^2)\gamma^4 + \\ & (1060\alpha^3 + 8828\alpha^2\beta + 8828\alpha\beta^2 + 1060\beta^3)\gamma^3 + \\ & (635\alpha^4 + 8828\alpha^3\beta + 17010\alpha^2\beta^2 + 8828\alpha\beta^3 + 635\beta^4)\gamma^2 + \\ & (110\alpha^5 + 2070\alpha^4\beta + 8828\alpha^3\beta^2 + 8828\alpha^2\beta^3 + 2070\alpha\beta^4 + 110\beta^5)\gamma + \\ & (5\alpha^6 + 110\alpha^5\beta + 635\alpha^4\beta^2 + 1060\alpha^3\beta^3 + 635\alpha^2\beta^4 + 110\alpha\beta^5 + 5\beta^6)\end{aligned}$$

We have to find all common points of a pair of curves ξ_{ij}, and calculate the corresponding polynomials U_{ij}.

ξ_{10}, ξ_{01} and ξ_{00}, ξ_{11}, respectively, cannot have a common point (α, β, γ) that satisfies (11). This can be simply verified by computing the resultant of the corresponding polynomials with respect to γ.

Since all linear combinations of the polynomials U_{10} and U_{11} contain the factor x_0, the number of nodal domains of such harmonic polynomials are already estimated in cases (ii), (iii) or (vi). Thus all common points of ξ_{10}, ξ_{11} and ξ_{01}, ξ_{11}, respectively, are not of interest for us any more.

Therefore computing the common points of ξ_{00} and ξ_{10} remain to estimate. (Because of symmetry we need not calculate the common points of ξ_{00} and ξ_{01}.) Their resultant is given by

$$\begin{aligned}R(\alpha,\beta) = \mathit{const}\cdot\beta^2\,(10\alpha+\beta)^4(16\alpha^2+16\alpha\beta+\beta^2)^2 \\ (-105\alpha^8 - 1260\alpha^7\beta + 15370\alpha^6\beta^2 + 18972\alpha^5\beta^3 - 186993\alpha^4\beta^4 \\ - 392840\alpha^3\beta^5 - 124248\alpha^2\beta^6 + 79488\alpha\beta^7 - 5120\beta^8)\end{aligned}$$

The only roots, which maybe satisfy (11) are the roots of the irreducible homogeneous factor of degree 8. Because of (11) we can assume $\beta = 1$. Thus we have to find all positive roots of

$$\begin{aligned}-5120 + 79488\alpha - 124248\alpha^2 - 392840\alpha^3 - 186993\alpha^4 \\ +18972\alpha^5 + 15370\alpha^6 - 1260\alpha^7 - 105\alpha^8 = 0\end{aligned} \tag{12}$$

Unluckily the roots of this polynomial cannot be calculated algebraically. By Newton's method one can get a very good approximation for all 8 of these. We pick out the positive root $\alpha_1 = 0.0755371323178$. Since $\beta_1 = 1$ by assumption, we can calculate γ_1 and find only one negative value: $\gamma_1 = -1.977147882$. We denote the polynomial of degree 2 with these coefficients with Q_1.

Now use (10) to calculate the other coefficients of

$$U_{10} = Q_1 \cdot P_{10} = x_0 x_2(\alpha x_0^2 + \beta x_1^2 + \gamma x_2^2)(\alpha' x_0^2 + \beta' x_1^2 + \gamma' x_2^2)$$
$$U_{00} = Q_1 \cdot P_{00} = (\alpha x_0^2 + \beta x_1^2 + \gamma x_2^2)(a x_0^4 + b x_1^4 + c x_2^4 + d x_0^2 x_1^2 + e x_0^2 x_2^2 + f x_1^2 x_2^2)$$

and get

$$\begin{array}{lll} \alpha_1' = 0.01201884918 & \beta_1' = 0.99388037056 & \gamma_1' = -0.10980599374 \\ a_1 = -0.56062745441 & b_1 = -0.04344483995 & c_1 = -0.03058148292 \\ d_1 = 0.64205475765 & e_1 = 0.51508373463 & f_1 = -0.07299733311 \end{array}$$

Figure 4 shows the nodal lines of the resulting functions in stereographic projection.

FIGURE 4: NODAL LINES OF u_{00} (LEFT HAND SIDE) AND u_{10} (RIGHT HAND SIDE) IN STEREOGRAPHIC PROJECTION OF THE NORTHERN HEMISPHERE (THE CROSS MARKS THE POLE).

Now we define the pencil $U(t) = t\, U_{10} + (1-t)\, U_{00} = Q_1 \cdot \big(t\, P_{10} + (1-t)\, P_{00}\big)$, $t \in [0, 1]$,

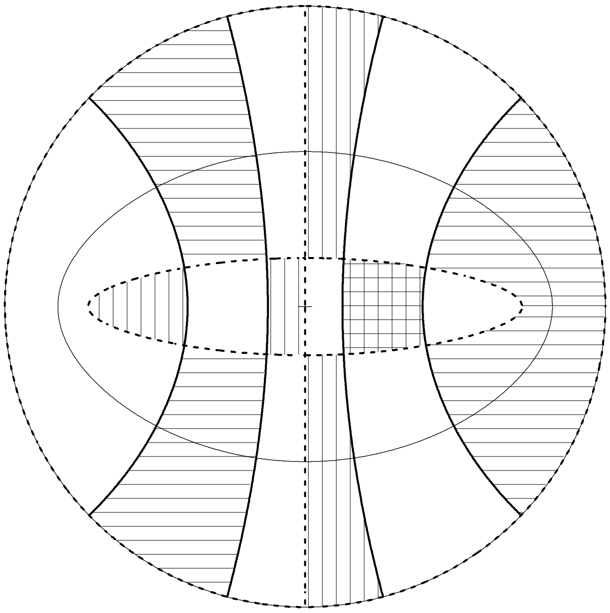

FIGURE 5: p_{00} (THICK LINE), p_{10} (DASHED LINE) (INCLUDES THE EQUATOR) AND THE COMMON CONIC q_1 OF THE PENCIL (THIN LINE) IN STEREOGRAPHIC PROJECTION OF THE NORTHERN HEMISPHERE.

of harmonic homogeneous polynomials. All polynomials in this pencil have the factor Q_1 and are not symmetric with respect to axes of q_1.

Figure 5 shows p_{00} (thick line), p_{10} (dashed line) and the common conic q_1 of the pencil (thin line) in stereographic projection of the northern hemisphere. The area, where p_{00} and p_{10} have different sign is hatched. First we notice that $\mu(u(0)) = \mu(u_{00}) = 18$. Since the nodal set of $u(t)$ changes continuously with increasing t, we can estimate the number of nodal domains for all spherical harmonics in this pencil (cf. also "the small perturbation theorem" in [GU] or [VO]).

Now let $\lambda(D)$ be the least eigenvalue of the Dirichlet eigenvalue problem on the domain D. Then the variational principle states, that for any two domains $D \subset\subset D'$, $\lambda(D) > \lambda(D')$. By means of this theorem we can deduce for each $t \in [0, 1]$ that

- there cannot be any singular point in the domains which are hatched vertically.
- there is at most one singular point, which must lie on the conic q, for each domain,

which is hatched horizontally. These cannot increase the number of nodal domains.

- there is at most one singular point in the two areas which are hatched both vertically and horizontally. These may increase the number of nodal domains by 2.
- there are no nodal lines within the unhatched domains, since P_{10} and P_{00} have the same sign.

Thus $\mu(u(t))$ cannot exceed $\mu(u(0)) + 2 = 20 < 24$. The above method does not need the exact location of the nodal lines of u_{10} and u_{00}. So we do not loose information if we use the approximate coefficients of these polynomials.

In the same way we can proceed with all positive roots of (12). In all these cases we have pencils of spherical harmonics of which we can show that the number of nodal domains cannot exceed 24.

This finishes the proof of our Theorem for the case $\ell = 6$. The estimates for cases $\ell = 4$ and $\ell = 5$ can be derived in the same way. Thus we have proved our Theorem.

Appendix: Proof of Proposition 2

By a proper choice of the coordinate system P can be written as

$$P = \alpha x_0^2 + \beta x_1^2 - \gamma x_2^2$$

where

$$\alpha \geq \beta > 0, \; -\gamma < 0 \tag{1}$$

are the eigenvalues of A_P with eigenvectors $e_1 = (0,0,1)$, $e_2 = (0,1,0)$, $e_3 = (1,0,0)$, and

$$Q = ax_0^2 + bx_1^2 + cx_2^2 + dx_0x_1 + ex_0x_2 + fx_1x_2$$

with real coefficients. In the following λ_1, λ_2 and λ_3 always denote the eigenvalues of A_Q. By assumption Q is irreducible.

Now if p, q have four common pairs of points and if their tangents are orthogonal at these, then the polynomial of degree two

$$R = \tfrac{1}{2}\nabla P \cdot \nabla Q = 2\alpha a x_0^2 + 2\beta b x_1^2 - 2\gamma c x_2^2 + (\alpha+\beta)dx_0x_1 + (\alpha-\gamma)ex_0x_2 + (\beta-\gamma)fx_1x_2$$

must be zero at these points also. Since p and q are different, they determine a pencil of conics, which contains r ([BE], Theorem 16.4.8, p. 185), i.e.

$$R = \mu P + \nu Q$$

for a proper choice of $\mu, \nu \in \mathbf{R}$. By comparison of coefficients we have

$$\begin{aligned} \mu\alpha + \nu a &= 2\alpha a & \qquad \nu d &= (\alpha+\beta)d \\ \mu\beta + \nu b &= 2\beta b & \nu e &= (\alpha-\gamma)e \\ -\mu\gamma + \nu c &= -2\gamma c & \nu f &= (\beta-\gamma)f \end{aligned} \tag{2),(3}$$

First we show, that there is at least one point which is a stationary point of both p and q, i.e. one of the eigenvectors of e_1, e_2 or e_3 von A_P must be eigenvector of A_Q. Since

$$A_Q = \begin{pmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{pmatrix}$$

it remains to show

Lemma 1:

At least two of the parameters d, e, f must be zero.

Proof:

Indirect: Suppose at most one of the parameters is zero.

Case: $d \neq 0, e \neq 0$ or $d \neq 0, f \neq 0$

In the first case we obtain from (3) that $\nu = \alpha + \beta$ and $\nu = \alpha - \gamma$ hence $\beta = -\gamma$, a contradiction to (1).

Analogously we obtain in the second case $\alpha = -\gamma$, again a contradiction to (1).

Case: $d = 0,\ e \neq 0, f \neq 0$

Again we obtain from (3) $\alpha = \beta$ and hence from (2)

$$0 = a(\nu - 2\alpha) + \alpha\mu$$
$$0 = b(\nu - 2\beta) + \beta\mu = b(\nu - 2\alpha) + \alpha\mu$$

and

$$a(\nu - 2\alpha) = b(\nu - 2\alpha)$$

Now either $\nu = 2\alpha = \alpha - \gamma$ (from (3)) and hence $\alpha = -\gamma$, a contradiction, or $a = b$.

Thus both p and q must be circles, which intersect in four points orthogonally. But since a circle is uniquely determined by three points, $p = q$, a contradiction.□

Next we show that this common stationary point is the center of one conic and the saddle point of the other.

First we deal with the case where all stationary points of p and q coincide, i.e.

Case: $d = e = f = 0$

Then Q reduces to

$$Q = ax_0^2 + bx_1^2 + cx_2^2$$

Lemma 2:

q is uniquely determined by p and a common point $(s, t, 1) \in p$.

Proof:

Suppose there would be another conic $q' \neq p$ with the same properties as q. The four points $(\pm s, \pm t, 1)$ are then common to p, q and q' and hence p must be contained in the pencil determined by q and q'. Since q and q' are tangent to p so every conic is contained in this pencil, especially this must be true for p itself, a contradiction.□

Thus Q can be uniquely written as

$$Q = \begin{vmatrix} x_0^2 & x_1^2 & x_2^2 \\ s^2 & t^2 & 1 \\ \alpha s^2 & \beta t^2 & -\gamma \end{vmatrix}$$

Hence we have from (1)

$$\begin{aligned} a &= t^2(\beta+\gamma) \ > 0 \\ b &= s^2(-\gamma-\alpha) \ < 0 \\ c &= s^2 t^2(\alpha-\beta) \ > 0 \end{aligned}$$

As can easily be seen, $\alpha \neq \beta$. Otherwise $c = 0$ and hence q would be singular and decomposable. Therefore p owns a saddle point, namely $e_2 = (0,1,0)$ and it must be the center of q. By changing the role of p and q we obtain in this case, that the center of p is the saddle point of q, too. This finishes the proof of Lemma 2.

Now all the cases remain, where p and q have only one common stationary point.

Case: $d \neq 0$ and $e = f = 0$

Then $e_1 = (0,0,1)$ — the center of p — is eigenvector of A_Q.

We show: e_1 is the saddle point of q.

From (2) and (3) we arrive at

$$\begin{aligned} \nu &= \alpha+\beta \\ \mu\alpha &= (\alpha-\beta)a \\ \mu\beta &= (\beta-\alpha)b \\ \mu\gamma &= (2\gamma+\alpha+\beta)c \end{aligned}$$

Since $(2\gamma+\alpha+\beta) > 0$, $\mu \neq 0$. Otherwise $c = 0$ and since c is the eigenvalue to the eigenvector e_1 (see below), Q would be decomposable. Moreover $\alpha \neq \beta$, else $\mu = 0$. We have from (1)

$$\begin{aligned} a &= \frac{\alpha}{\alpha-\beta} \ > 0 \\ b &= \frac{\beta}{\beta-\alpha} \ < 0 \\ c &= \frac{\gamma}{2\gamma+\alpha+\beta} \ > 0 \end{aligned} \tag{4}$$

The characteristic polynomial of A_Q is

$$\left((a-\lambda)(b-\lambda)-\tfrac{1}{4}d^2\right)(c-\lambda) = 0$$

and hence the eigenvalues are

$$\begin{aligned}
\lambda_2 &= c \\
\lambda_{1,3} &= \tfrac{a+b}{2} \mp \sqrt{(\tfrac{a+b}{2})^2 - ab + \tfrac{1}{4}d^2} \\
&= \tfrac{1}{2}\left((a+b) \mp \sqrt{(a-b)^2 + d^2}\right) \\
&= \tfrac{1}{2}\left(1 \mp \sqrt{\left(\tfrac{\alpha+\beta}{\alpha-\beta}\right)^2 + d^2}\right)
\end{aligned}$$

Since $(a-b)^2 > (a+b)^2$, $\lambda_1 < 0$. Since $\alpha+\beta > 0$, $c = \frac{\gamma}{2\gamma+\alpha+\beta} < \frac{1}{2}$, and hence $\lambda_2 = c < \lambda_3$. Therefore, the eigenvalues of A_Q are different, i.e. the saddle point of q exists, and (since λ_2 is the eigenvalue to the eigenvector e_1) this saddle point is e_1, the center of p as proposed. Moreover together with (4) we find, that neither p nor q is a circle.

Case: $e \neq 0$, $d = f = 0$

Then $e_2 = (0,1,0)$ — the saddle point of p — is eigenvector of A_Q.

We show: e_2 is the center of q.

Completely analogously to the above case we have

$$2\beta - \alpha + \gamma \neq 0$$

and

$$\begin{aligned}
a &= \frac{\alpha}{\alpha+\gamma} > 0 \\
b &= \frac{\beta}{2\beta-\alpha+\gamma} \\
c &= \frac{\gamma}{\gamma+\alpha} > 0
\end{aligned} \tag{5}$$

We show:

$$b < 0 \tag{6}$$

$$e^2 < 4\frac{\alpha\gamma}{(\alpha+\gamma)^2} \tag{7}$$

From the characteristic polynomial of A_Q we have

$$\begin{aligned}
\lambda_1 &= b \\
\lambda_{2,3} &= \frac{1}{2}\left(1 \pm \sqrt{\frac{(\alpha-\gamma)^2}{(\alpha+\gamma)^2} + e^2}\right)
\end{aligned}$$

Because of (7) the following holds

$$\frac{(\alpha-\gamma)^2}{(\alpha+\gamma)^2} + e^2 < \frac{(\alpha-\gamma)^2}{(\alpha+\gamma)^2} + 4\frac{\alpha\gamma}{(\alpha+\gamma)^2} = \frac{(\alpha+\gamma)^2}{(\alpha+\gamma)^2} = 1$$

Hence the eigenvalue $\lambda_1 = b$ to the eigenvector e_2 is negative and both other eigenvalues are positive. Thus e_2 is the center of q. For completeness we have to show that e_2 really is the saddle of p, i.e. $\alpha \neq \beta$. But this immediately follows if we change the roles of p and q.

To show that $b < 0$ we calculate the resultant of P and Q with respect to y and show that the common points of p and q would be complex otherwise. We get

$$\operatorname{Res}(x_0, x_2) = \begin{vmatrix} \alpha x_0^2 - \gamma x_2^2 & 0 & \beta & 0 \\ 0 & \alpha x_0^2 - \gamma x_2^2 & 0 & \beta \\ a x_0^2 + e x_0 x_2 + c & 0 & b & 0 \\ 0 & a x_0^2 + e x_0 x_2 + c & 0 & b \end{vmatrix}$$

$$= \left((a\beta - \alpha b)x_0^2 + e\beta x_0 x_2 + (\beta c + b\gamma)x_2^2\right)^2$$

The term $(a\beta - \alpha b)$ must not vanish. Otherwise two of the common points would be on the line $x_2 = 0$, a contradiction. We can assume $x_2 = 1$. Then the x_0-components of the common points of p and q are the solutions of the quadratic equation

$$(a\beta - \alpha b)x_0^2 + e\beta x_0 + (\beta c + b\gamma) = 0$$

which are

$$x_{0\pm} = \frac{-e\beta \pm \sqrt{e^2\beta^2 - 4(a\beta - \alpha b)(\beta c + b\gamma)}}{2(a\beta - \alpha b)}$$

These must be in the real interval $\left[-\sqrt{\frac{\gamma}{\alpha}}, \sqrt{\frac{\gamma}{\alpha}}\right]$, otherwise the y-component of the common points would not be real, which can be seen from the definition of P.

Lemma 3:

If $x_{0\pm} = \sigma\sqrt{\frac{\gamma}{\alpha}}$, with $\sigma = \pm 1$, then $\sigma e = -2\frac{\sqrt{\alpha\gamma}}{\alpha+\gamma}$

Proof:

We transform and square the radical equation

$$\frac{-e\beta \pm \sqrt{e^2\beta^2 - 4(a\beta - \alpha b)(\beta c + b\gamma)}}{2(a\beta - \alpha b)} = \sigma\sqrt{\frac{\gamma}{\alpha}}$$

and get

$$e^2\beta^2 - 4(a\beta - \alpha b)(\beta c + b\gamma) = 4(a\beta - \alpha b)^2\,\sigma^2\frac{\gamma}{\alpha} + e^2\beta^2 + 4(a\beta - \alpha b)\,\sigma\sqrt{\frac{\gamma}{\alpha}}\,e\beta$$

and

$$\sigma e = \frac{1}{\beta}\left(-\sqrt{\frac{\gamma}{\alpha}}(a\beta - \alpha b) - \sqrt{\frac{\alpha}{\gamma}}(\beta c + b\gamma)\right)$$

From (5) we have $(a\beta - \alpha b) = \frac{\alpha\beta}{\alpha+\gamma} - \frac{\alpha\beta}{2\beta-\alpha+\gamma}$ and $(\beta c + b\gamma) = \frac{\beta\gamma}{\gamma+\alpha} + \frac{\beta\gamma}{2\beta-\alpha+\gamma}$. Using this the result follows.□

Lemma 4:

$2\beta - \alpha + \gamma < 0$ if and only if $a\beta - \alpha b > 0$.

Proof:

By assumption (1) $\alpha \geq \beta$ and thus $2\beta - \alpha + \gamma \leq \alpha + \gamma$. Now if $2\beta - \alpha + \gamma < 0$ then $\frac{1}{\alpha+\gamma} \geq \frac{1}{2\beta-\alpha+\gamma}$ and because of (5) we have $(a\beta - \alpha b) = \alpha\beta\left(\frac{1}{\alpha+\gamma} - \frac{1}{2\beta-\alpha+\gamma}\right) \geq 0$ In the same way we yield $(a\beta - \alpha b) \leq 0$ if $2\beta - \alpha + \gamma > 0$. Since $(a\beta - \alpha b) \neq 0$ the result follows.

Lemma 5:

$(a\beta - \alpha b) > 0$.

Together with Lemma 4 we get $2\beta - \alpha - \gamma < 0$ and hence $b < 0$ as proposed.

Proof:

Suppose $(a\beta - \alpha b) < 0$. Then the map $e \mapsto x_{0-}(e) - \sqrt{\frac{\gamma}{\alpha}}$ with $x_{0-}(e) = \frac{-e\beta - \sqrt{e^2\beta^2 - 4(a\beta-\alpha b)(\beta c + b\gamma)}}{2(a\beta - \alpha b)}$ would become positive for sufficiently large e. By Lemma 3 the only zero of this function is $e = -2\frac{\sqrt{\alpha\gamma}}{\alpha+\gamma}$. Thus $e \leq -2\frac{\sqrt{\alpha\gamma}}{\alpha+\gamma} < 0$ if $x_{0-}(e) \leq \sqrt{\frac{\gamma}{\alpha}}$. In the same way we have $e \geq 2\frac{\sqrt{\alpha\gamma}}{\alpha+\gamma} > 0$ if $x_{0+}(e) \geq -\sqrt{\frac{\gamma}{\alpha}}$, a contradiction. Since $(a\beta - \alpha b) \neq 0$ the proposition follows.□

In the same way we have for $(a\beta - \alpha b) > 0$, $e^2 < 4\frac{\alpha\gamma}{(\alpha+\gamma)^2}$, which is statement (7).

Case: $f \neq 0$ and $d = e = 0$
We show: This case is not possible.

Again we have from (2), (3) and (1) $a = \frac{\alpha}{2\alpha-\beta+\gamma} > 0$, $b = \frac{\beta}{\beta+\gamma} > 0$ and $c = \frac{\gamma}{\beta+\gamma} > 0$, i.e. q does not have any real points.

This finishes the proof.$\square$

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