Nodal Properties
of
Spherical Harmonics

Dissertation

zur Erlangung des akademischen Grades

Doctor rerum naturalium

vorgelegt von

Mag. Josef Leydold

an der Formal- und Naturwissenschaftlichen Fakultät der

Universtität Wien

im Jänner 1993

za
Mojco
in
Mirana

[. . .] die Grundlagen fehlen, es ist alles fragwürdig und zweifelhaft geworden, alles relativiert und durchwühlt, und dennoch besteht jener Trieb zu Arbeit und Spiel ungebrochen fort, und wie wir Künstler, so bemühen sich auch die Männer der Wissenschaft eifrig weiter, ihre Beobachtungswerkzeuge und ihre Sprache zu verfeinern und dem Nichts oder Chaos wenigstens einige sorgfältig beobachtete und beschriebene Aspekte abzugewinnen.

Hermann Hesse
in: Der Bettler

Zusammenfassung

In der vorliegenden Arbeit werden die Knotenlinien und Knotengebiete von Kugelflächenfunktionen untersucht. Die Motivation, sich gerade mit den Lösungen dieser speziellen elliptischen partiellen Differentialgleichung zu beschäftigen, liegen zum einen darin, in der relativen Einfachheit des Problems (die Kugelflächenfunktionen sind harmonische homogene Polynome eingeschränkt auf die Kugeloberfläche; die Sphere is kompakt; u.ä.) zum anderen in der großen Bedeutung dieser Funktionen für allgemeine elliptische Differentialgleichungen. Trotzdem gibt es bis heute noch viele ungelöste Probleme.

Im ersten Kapitel werden einige bekannte Eigenschaften der Kugelflächenfunktionen zusammengefaßt und eine Vermutung für eine scharfe obere Schranke für die Anzahl der Knotengebiete gemacht.

Im zweiten Teil wird versucht, die Sätze über ebene reelle algebraische Kurven auf die Knotenlinien der Kugelflächenfunktionen anzuwenden. Dabei stellt sich heraus, daß mit diesen algeraischen Methoden zunächst die gleichen Resultate erzielt werden wie mit dem bereits seit längerem bekannten Satz von Courant. Durch Verwendung weiterer Eigenschaften (z.B. des Satzes von Bers) läßt sich jedoch eine Verbesserung erzielen. Genauer: Es ist möglich eine Klassifizierung aller Kugelflächenfunktionen, die vollständig in Linearfaktoren zerfallen, vorzunehmen; man kann eine obere Schranke an eine große Anzahl von Kugelflächenfunktionen, die nur aus quadratischen und linearen Faktoren bestehen, herleiten. Darüber hinaus kann noch eine Reihe von Eigenschaften von Kugelflächenfunktionen hergeleitet werden, um die anfangs gemachte Vermutung für alle Kugelflächenfunktionen von Ordnung kleiner oder gleich sechs zu beweisen.

Im dritten Teil wurde eine obere Schranke für die Anzahl der Knotengebiete von nicht-singulären Kugelflächenfunktionen hergeleitet, die besser ist, als jene, die sich aus dem Satz von Harnack ergibt. Es konnte dadurch gezeigt werden, daß es keine harmonischen (Harnackschen) M-Kurven geben kann. Weiters wurde durch Konstruktion gezeigt, daß die gefundene Schranke in der richtigen Ordnung vom Grad der Kugelflächenfunktionen abhängt.

Im vierten Teil wird versucht mittels Graphen “Knotenmuster” zu beschreiben, bzw. diesen Begriff überhaupt erst mathematisch zu erfassen.

Im letzen Teil wurde versucht, den Raum der Kugelflächenfunktionen zu beschreiben. Besonderes Interesse galt dabei der Größe der einzelnen Komponenten der Menge aller normierten nicht-singulären Kurven. Dabei wurden auch einige Ungleichungen über Kugelflächenfunktionen hergeleitet.

Abstract

In the following we study the properties of nodal lines and nodal domains of spherical harmonics. The motivation for dealing with the solutions of this special elliptic PDE is on the one hand that this problem is relatively simple (the spherical harmonics are harmonic homogeneous polynomials restricted onto to the sphere; the sphere is compact; etc.), on the other hand the importance of these functions for arbitrary PDE. In spite of of this a lot of problems are open till now.

In the first chapter we summarize some well-known properties of spherical harmonics and make a conjecture on a sharp estimate for the number of nodal domains.

In the second part we apply theorems about plane real algebraic curves on nodal lines of spherical harmonics. It turns out, that we arrive at the same results achieved by Courant's theorem. By using further properties (e.g. by Bers' theorem) we get better results. In detail: It is possible to make a full classification of spherical harmonics, which decomposes into linear factors only; we can derive a sharp upper bound for the number of nodal domains for a large class of spherical harmonics, which consists of linear and quadratic factors only. Moreover some further properties of spherical harmonics can be found, such that we can proof the above conjecture for degree less or equal to six.

In the third part an upper bound for the number of nodal domains of non-singular spherical harmonics is derived, which is better than Harnack's theorem. Hence there are no harmonic (Harnack) M-curves. Furthermore it is shown by construction, that this found estimate is of right order.

In the forth part graphs are used to describe and define the conception of "nodal pattern".

In the last part the space of spherical harmonics — especially the components of the set of normalized non-singular curves — are studied. Moreover a few inequalities about spherical harmonics are deduced.

Table of Contents

Zusammenfassung (german abstract) .... *i*
Abstract .... *iii*
Preface .... *v*
Spherical Harmonics 1
Algebraic curves 10
Straight lines 22
Conics 32
A few sharp bounds 47
Non-singular curves 63
Nodal Pattern 72
The eigenspace 77
Appendix A 97
References 110
Some used symbols 114

Preface

The properties of the nodal set of solutions of eigenvalue problems

$$((-\Delta + V(x))\, u = \lambda u$$

on smooth Riemannian manifolds are subject of many investigations. In the last years some results have been proven (e.g. estimates on the length of nodal lines in [DF], [HS] or [KR]). A survey of the development (until 1984) is given in [CI]. But a lot of problems are left open till now.

In this paper we study spherical harmonics[1], the solution of the eigenvalue problem

$$-\Delta u(x) = \lambda u(x) \quad \text{for all } x \in S^2$$

The reasons for choosing this special problem are the importance of the spherical harmonics on the one hand and some technical aspects, which allow us the use of very different methods such as the variational principle and the theory of algebraic curves, on the other hand. In detail:

- (•) The spherical harmonics are explicitly know.
- (•) They are the restriction of harmonic homogeneous polynomials to the sphere.
- (•) The 2-sphere is compact, has no boundary and is invariant under revolution, i.e. it is invariant under the action of the $SO(3)$.
- (•) Solutions of arbitrary eigenfunctions look locally like harmonic homogeneous polynomials at their nodal sets (see [CS], [HO1] and [HO2])

Figure 1 shows the nodal lines of some spherical harmonics.
Nevertheless a lot of open problems arise. For example there is no sharp upper bound for the number of nodal domains of a spherical harmonic, i.e. the number of components of the complement of its nodal set (see [AR3]). Solving this problem is the starting point of the following work.

In the first chapter we give a short survey of well-known results and state a few additional lemmata (Euler's theorem for spherical harmonics, the orange slice argument). These are preliminary results, which we either need for our work or are cited here to illustrate our improvements (e.g. the theorems of Courant, Pleijel).

[1] For definition an details about spherical harmonics see the next chapter

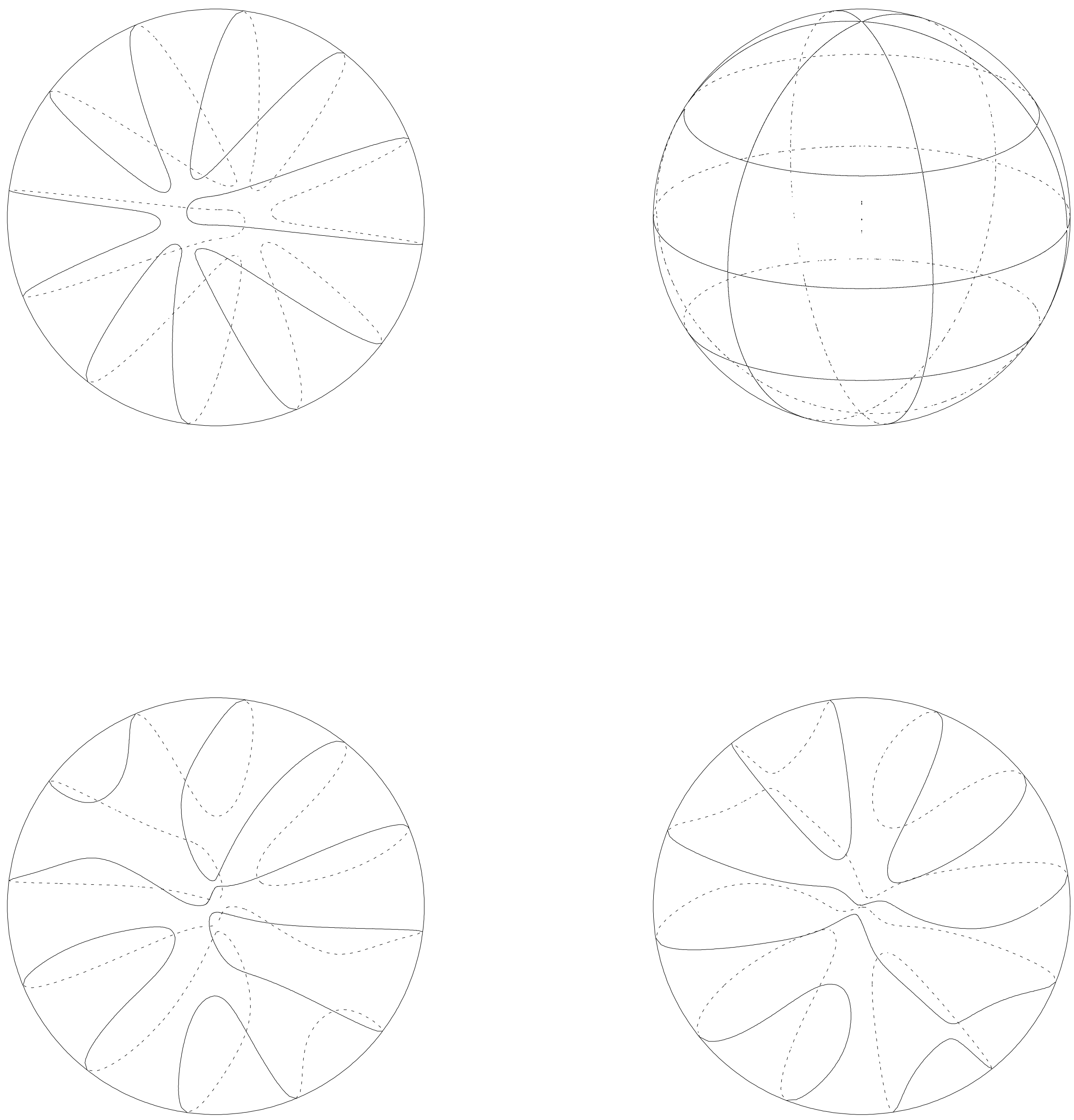


Figure 1

In the second part we use the theorem and methods of (real) plane algebraic curves to derive some results on the nodal set of spherical harmonics. Our main tools are the theorems of Bezout, Harnack (for non-singular real algebraic curves) and Bers and the resultant of two polynomials. Curiously in a first step the same estimate follow, which are already known from Courant's theorem. Using Bers' theorem this estimate can be improved a little bit (see [KA2]). But this new bound is not sharp, either. But by using some tricks we are able to derive such a sharp upper bound the number of nodal domains for spherical harmonics of degree ≤ 6, which is better than Courant's theorem.
Besides this we make a full classification of spherical harmonics which decomposes into linear factors only. These functions are closely related to the finite group of revolution and hence to the Platonic solids. We also can give a sharp upper bound for the number of nodal domains of almost all spherical harmonics, which consists of linear and quadratic factors only.
It is obvious that these results cannot be extended to dimensions greater than 2 easily, since then the theorems of Harnack and Bezout are not in such a suitable form.

Besides these attempts to improve Courant's theorem, trials are made to find an estimate for the number of nodal domains of non-singular spherical harmonics, i.e. of spherical harmonics which have no singular points (points, where the function and the gradient vanish). By means of the "orange slice argument" we can derive a bound, which is better than that derived from Harnack's theorem. Thus we are able to show, that there are no harmonic M-curves. (Harnack deduced a sharp upper bound for the number of components of non-singular real plane algebraic curves. Curves with the maximum number of components are called M-curves.)

Another starting point of this problem is the following: At first we search for a lower bound for the "size" of the components of the set of non-singular spherical harmonics in the eigenspace to a given eigenvalue. The nodal sets of spherical harmonics from the same component have the same isotopy type and especially the same number of nodal domains. Secondly, if we have found such a bound, we select normalized non-singular spherical harmonics from the eigenspace, study their nodal lines and count their nodal domains. The bound of the first step ensures us, that we find all possible isotopy types of non-singular spherical harmonics for a given eigenvalue by this method.

In the chapter about the eigenspace a function, that gives the "size" of these component, is derived. This function gives the (euclidean) distance between a given spherical harmonic and the set of singular spherical harmonics. Although properties of this "distance" function are derived by a method due to R. Sperb (by means of the maximum principle), our goal to estimate the least local maximum cannot be arrived. Thus this chapter contains some estimates on the global maximum of this function and a few more results that can be deduced by means of it.

A last attempt is made by means of graphs. In this concept (an idea of T. Hoffmann-Ostenhof) graphs are used to describe the location of the nodal domains of spherical harmonics. Restrictions on these graphs are given.

For some calculations (for example for resultants), especially for checking the equivalence of two expressions, computer algebra (Mathematica 2.0) was used. For the plots of the nodal lines of the base functions Y_ℓ^m the locations of the meridians and latitude circles where computed and then plotted by means of PARAMETRICPLOT[]. The plots of the nodal lines of the other spherical harmonics are made by means of the graphic package "ERLGRAPH 2.10M" on a SUN workstation.

An dieser Stelle möchte ich mich für die Hilfe all derer bedanken, die zum Entstehen dieser Arbeit beigetragen haben.
Bei Thomas Hoffmann-Ostenhof für die interessante Fragestellung und für die vielen Diskussionen und Anregungen. Bei Harald Rindler dafür, daß er kurzfristig als Zweitbegutachter eingesprungen ist.
Bei allen Kollegen und Kolleginnen am Institut für theoretische Chemie der Universität Wien, sowie am Institut für Statistik der WU Wien für zahlreiche Hilfestellungen und das gute Arbeitsklima, insbesondere bei Jörg Lenneis, Wolgang Hörmann, Christian Forst und Peter Stadler.
Herzlich bedanken möchte ich mich bei meinen Eltern, die mir dieses Studium erst ermöglichten.
Bei meiner Frau Mojca für die moralische Unterstützung meiner Arbeit und die große Geduld die sie dabei bewiesen hat. Bei meinem Sohn Miran für das aufmerksame Zuhören beim Vorlesen meiner Arbeit sowie für seine Verkostung derselben.
Bei Herrn Mag. Peter Fuchs im voraus für die sorgfältige Beschlagwortung in der Nationalbibliothek.

Ich möchte mich auch dafür bedanken, daß mir vom Institut für theoretische Chemie der Universität Wien und vom Institut für Statistik, Abteilung für angewandte Statistik und Datenverarbeitung, der WU Wien sowohl Arbeitsplatz als auch -gerät zur Verfügung gestellt worden sind.

Spherical Harmonics

The eigenvalue problem

As is well-known the *spherical harmonics* are the solution of the closed eigenvalue problem on the two-sphere

$$(-\Delta + \lambda)u = 0 \quad \text{on } S^2 \tag{1}$$

The eigenvalues are $\lambda = \ell(\ell+1)$, $\ell \in \mathbb{N}_o$. The corresponding eigenspaces are degenerated and have dimension $(2\ell+1)$. We want to use the following base functions for the eigenspace to the eigenvalue $\lambda = \ell(\ell+1)$

$$Y_\ell^m(\theta,\varphi) = c_{\ell m} \begin{cases} P_\ell^m(\cos\theta)\cdot\sin m\varphi & \text{if } 1 \le m \le \ell \\ P_\ell(\cos\theta) & \text{if } m = 0 \\ P_\ell^{|m|}(\cos\theta)\cdot\cos|m|\varphi & \text{if } -\ell \le m \le -1 \end{cases}$$

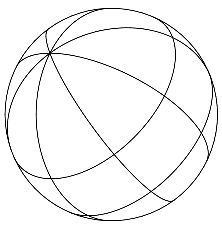
where P_ℓ^m denotes the associated Legendre polynomials and

$$c_{\ell m} = \sqrt{\frac{(\ell-|m|)!\,(2\ell+1)}{(\ell+|m|)!\,4\pi}} \cdot \begin{cases} 1 & \text{if } m = 0 \\ \sqrt{2} & \text{otherwise} \end{cases}$$

are normalizing factors. (θ and φ denote the spherical coordinates.) Obviously these base functions have the following property (see figure 1)

Lemma 1:

The nodal set of Y_ℓ^m consists of m meridians and $(\ell-m)$ latitude circles, which are symmetric with respect to the equator. Therefore they have $2m(\ell-m+1)$ nodal domains.



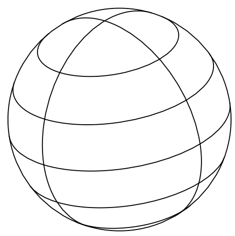


Figure 1: The nodal lines of Y_6^4 and Y_7^2.

The functions $Y_\ell^0 = P_\ell(\cos\theta)$ are called *zonal harmonics* and their nodal sets consist of ℓ latitude circles. Thus they are invariant under rotation about the axis through the poles. Their main properties can be found in [SW], p. 143ff.

Each eigenfunction u to the eigenvalue $\lambda = \ell(\ell+1)$ can be uniquely written in terms of the functions Y_ℓ^m, i.e.

$$u(\theta,\varphi) = \sum_{i=-\ell}^{\ell} \alpha_i Y_\ell^i(\theta,\varphi)$$

for a proper set $\{\alpha_{-\ell},\ldots,\alpha_0,\ldots,\alpha_\ell\}$.

These eigenfunctions are the restrictions of *harmonic homogeneous polynomials* U in $\mathbb{R}^3$ of degree ℓ to the sphere S^2, i.e. of homogeneous polynomials of degree ℓ which satisfy

$$\Delta U = 0 \quad \text{on } \mathbb{R}^3 \tag{2}$$

There is a one to one relation between the spherical harmonics and harmonic polynomials via

$$U(x) = |x|^\ell \cdot u\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^3 \tag{3}$$

Therefore spherical harmonics and harmonic homogeneous polynomials are in this sense just two sides of the same medal.

Because U is a homogeneous polynomial of degree ℓ we immediately have

Lemma 2: (INVERSION(ANTI)SYMMETRY)

$$u(-x) = (-1)^\ell u(x)$$

Notations and conventions

We denote the nodal set u with $\mathcal{N}(u) = u^{-1}(0)$. If there is no risk of confusion we sometimes say simply u instead of $\mathcal{N}(u)$ (see also the remarks 1 and 2 on algebraic curves).

We always use ***capital letters*** (e.g. U or P) to denote homogeneous polynomials in $\mathbb{R}^3$ and ***small letters*** (e.g. u or p) to denote their restrictions to the sphere, e.g. $p = P|_{S^2}$. We call the restriction of an arbitrary homogeneous polynomial to the sphere a *spherical polynomial* (although this name might cause a confusion with the name "spherical functions" for spherical harmonics).

Y_ℓ^m always denotes the base functions of the spherical harmonics, $P_\ell^m(\cos\theta)$ the Legendre polynomials.

u and U always denote a spherical harmonic and a harmonic homogeneous polynomial of degree ℓ, respectively. The degree of u is defined as the degree of U. ℓ always means the degree of u and U.

Because of lemma 1 to each point of $\mathcal{N}(u)$ there exists exactly one antipodal point on the sphere which also is in $\mathcal{N}(u)$. Hence we always mean ***a pair of antipodal points*** if we say ***a point*** of the nodal set. But if we say *a component* of $\mathcal{N}(u)$ we mean just one set, ***not*** the pair of antipodal components.

Since the great circles are the geodesics on the sphere with the standard euclidean metric, we call them ***straight lines***. The straight lines (or great circles) through the pole of the coordinate system on the sphere are called *meridians*. Circles, of which the center is the pole, are called *latitude circles*. The only latitude circle, which is a straight line, is called *equator* and divides the sphere into two hemispheres.

A *sector with angle at x* is a twoangle (a polygon with two vertices) with vertices at x and $-x$. The angle is the angle between the two straight lines, which form the border of the sector, inside the twoangle.

Some indices

Let p be any spherical polynomial. Then we denote

$\mu(p) :=$ the number of nodal domains, i.e. the number of components of the complement of $\mathcal{N}(p)$.

$\varrho_p(x) :=$ the multiplicity of p at x (also called the order of vanishing of p at x).

$\nu_p(x) := \varrho_p(x) - 1$

$\nu(p) := \sum \nu_p(x)$, where the sum is taken over all singular points of p.

$\zeta(p) :=$ number of components of $\mathcal{N}(p)$

Preliminary results

In the following we want to summarize the results on spherical harmonics. Some of these exist in a more general form for a much greater class of eigenvalue problems.

First we need for simplicity for later use

Lemma 3:
The pole of the coordinate system of the sphere can be chosen arbitrary, i.e. any point of $\mathcal{N}(u)$ can be chosen as the pole.

Proof:
Since the Laplace-Beltrami operator on the sphere is invariant under rotation, the proposition follows.□

Lemma 4: (Unique continuation)
If there exists an open set G in S^2 where $u|_G = 0$, then $u \equiv 0$.

Proof:
Since U is a polynomial, this is trivial.□

The next results show us, that the topology of the nodal lines is rather simple.

Lemma 5: (Harnack's inequality)
Let D be a domain where u does not change sign. Then for every closed connected set $D' \subset\subset D$ there exists a constant $c > 0$ such that

$$\max_{x \in D'} |u(x)| \leq c \min_{x \in D'} |u(x)|$$

Proof:
(see [GT])

Corollary:
A spherical harmonic u must change sign at its nodal lines and hence its singular points must be isolated.

More information about the topology near this points we obtain from

Lemma 6: (Bers' theorem)
If k local branches of the nodal lines of u intersect in a point x, their tangents form an equiangular system. Moreover $\varrho_u(x) = k$, i.e. x is a point of u of multiplicity k.

Proof:
(see [CS])

Because of this nice property we can use the theorem about the Euler characteristic of the sphere to show a global result on nodal lines.

Lemma 7: (EULER'S THEOREM FOR SPHERICAL HARMONICS)
Let u be a spherical harmonic. Then

$$\mu(u) - \nu(u) - \zeta(u) = 1$$

Proof:
It is well known, that the nodal lines of spherical harmonics are smooth curves. Now if $\mathcal{N}(u)$ is connected, we obtain a cellular decomposition of the sphere, where the 2-cells are the nodal domains of u and the nodal lines form the 1-skeleton of this finite CW complex. We split each of the closed curves in this cellular complex, so that the closure of each 1-cell (i.e. the image of an injective curve) contains exactly 2 0-cells (i.e. the points of the cellular complex). For this we can use Eulers theorem (see e.g. [VJ], p. 73) to obtain

$$\#\text{ 0-cells} - \#\text{ 1-cells} + \#\text{ 2-cells} = \chi(S^2) = 2$$

Obviously $\mu(u) = \#$ 2-cells. From Bers theorem we know that the curves form an equiangular system of $\varrho_u(x)$ branches when they meet at a point x. So any 0-cell is in the closure of exactly $2\varrho_u(x)$ 1-cells. Since the closure of any 1-cell contains exactly 2 0-cells, we obtain that $\sum \varrho_u(x)$, taken over all 0-cells, is the number of 1-cells. Since $\nu_p(x) := \varrho_p(x) - 1$, we obtain $\nu(u) = \sum(\varrho_u(x) - 1) = \#$ 1-cells $- \#$ 0-cells, and the result follows.

If $\mathcal{N}(u)$ is not connected, we need exactly $(\zeta(u) - 1)$ injective lines, which join the $\zeta(u)$ components, to obtain a finite CW complex. Considering these supplementary lines, the result follows.□

Later we also need a similar result for arbitrary spherical polynomials. But since in this case cusps, isolated singular points, etc. can occur, only an inequality holds.

Lemma 7A: (EULER'S THEOREM FOR SPHERICAL POLYNOMIALS)
Let p be a spherical polynomial of degree ℓ without multiple factors (i.e. either p is irreducible or each factor appears only once). Then

$$\mu(p) - \nu(p) - \zeta(p) \leq 1$$

Proof:
The zero set of a spherical polynomial consists of distinct points and smooth one dimensional manifolds, which connect (some of) the points (see e.g. [WH]). So we again obtain a cellular complex. But in this case each 0-cell is in the closure of *at most* $2\varrho_p(x)$ 1-cells. So if p has no multiple factors, the result follows.□

From the next result we will deduce an usefull method.

Lemma 8: (VARIATION PRINCIPLE)
Let $D' \supseteq D$. Then $\lambda(D') \leq \lambda(D)$, where $\lambda(D)$ is the first eigenvalue of the Dirichlet eigenvalue problem on D. Equality holds if and only if $D' = D$.

To formulate the following method we need special sectors on the sphere. A *sectorial hull with angle at* x of a nodal domain or a closed curve is the intersection of all sectors with angle at x which contain the nodal domain or the closed curve, respectively. Its *angle* is the infimum over the angles of all these sectors.

Lemma 9: (THE ORANGE SLICE ARGUMENT)
Let D be any nodal domain of a spherical harmonic u. Then any sectorial hull S of D has angle greater than $\frac{\pi}{\ell}$ (see figure 2). If there exists such a sectorial hull with angle $\frac{\pi}{\ell}$, then $u = Y_\ell^\ell$.

Proof:
From lemma 1 we know that the nodal domains of Y_ℓ^ℓ are sectors S' with angle $\frac{\pi}{\ell}$. So if D has a sectorial hull with angle less then $\frac{\pi}{\ell}$, D would be a proper subset of such a domain S'. A contradiction to the variation principle.□

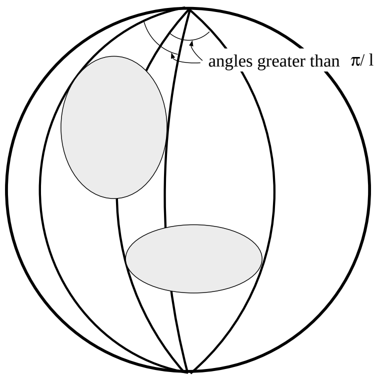


FIGURE 2: THE SECTORIAL HULLS OF TWO NODAL DOMAINS.

Estimates for the number of nodal domains

The first estimate for the number of nodal domains can be obtained from Courant's nodal domain theorem. This theorem is true for all Dirichlet and closed eigenvalue problems on a smooth manifold of arbitrary dimension. It is sharp for the one dimensional case.

Lemma 10: (Courant's nodal domain theorem)
Let ψ_k be an eigenfunction to the k-th eigenvalue (counted with multiplicity) of the Laplace-Beltrami-operator on a compact C^∞-manifold M (without boundary). Then

$$\mu(\psi_k) \leq k$$

Proof: ([CS])
Suppose ψ_k has the nodal domains $D_1, \ldots, D_k, D_{k+1}, \ldots$. We define $\Psi_i = \psi_k$ on D_i and $\Psi_i = 0$ outside D_i, for $1 \leq i \leq k$. We can find real numbers α_i such that $\Psi = \sum_{i=1}^{k} \alpha_i \Psi_i$ is orthogonal to the space generated by $\psi_1, \ldots, \psi_{k-1}$ and

$$\int_M |\Psi|^2 dx = 1 \tag{4}$$

Then we have

$$\lambda_k \leq \int_M |\nabla \Psi|^2 = \sum_{i=1}^{k} \alpha_i^2 \int_M |\nabla \Psi_i|^2 = \sum_{i=1}^{k} \alpha_i^2 \int_{D_i} |\nabla \psi_k|^2 \tag{5}$$

where λ_k is the k-th eigenvalue. However, ψ_k is the k-th eigenfunction. The nodal sets of ψ_k form a compact C^∞-manifold except on a closed set of lower dimension (see [CS]). Thus

$$\int_{D_i} |\nabla \psi_k|^2 = -\int_{D_i} \psi_k \Delta \psi_k = \lambda_k \int_{D_i} \psi_k^2 \tag{6}$$

Consequently from (5) and (6) we have

$$\int_M |\nabla \Psi|^2 = \lambda_k$$

Hence by min-max principle (see e.g. [CI], p. 17) and by the spectral theorem (for compact operators) Ψ itself is an eigenfunction (in L^2-sense). But $\Psi \equiv 0$ on the open set D_{k+1}, which implies $\Psi \equiv 0$ on M by the unique continuation of eigenfunctions, a contradiction to (4).□

Now we apply this theorem to spherical harmonics. Using the fact that the eigenvalue $\ell(\ell+1)$ of the Laplace-Beltrami operator on the sphere has multiplicity $2\ell+1$ we get by a short calculation

Theorem 1: (COURANT'S ESTIMATE, FIRST VERSION)
Let u be a spherical harmonic, then $\mu(u) \leq \ell^2 + 1$.

We can improve this result slightly by using the inversion(anti)symmetry of the spherical harmonics. Every symmetric L^2-function is orthogonal to any antisymmetric L^2-function on the sphere. So we can deal with symmetric and antisymmetric spherical harmonic separately and obtain in the same way as in proof of theorem 1

Theorem 1A: (COURANT'S ESTIMATE, SECOND VERSION)
Let u be a spherical harmonic. Then

$$\mu(u) \leq \ell(\ell-1) + 2$$

The proof of Courant's theorem is completely analytic (min-max principle, variation principle, unique continuation) using the fact that the spherical harmonics are the eigenfunctions of the Laplace-Beltrami operator. Later we will use the fact that they are the restriction of harmonic homogeneous polynomials in $\mathbb{R}^3$ to the sphere. For the homogeneous polynomials we can use Harnack's theorem (do not mix up with Harnack's inequality) to derive an estimate for the number of the components of the complement of an "algebraic curve" (its "nodal" domains). Curiously these bounds are the same.

Indeed, Bérard and Meyer ([BM]) have shown that equality in Courant's nodal domain theorem occur for at most a finite number of eigenvalues. They have used Pleijel's theorem (see [CI], p. 24) and the Faber-Krahn inequality (see [CI], p. 87), which states

Lemma 11: (FABER-KRAHN INEQUALITY)
Let D be a nodal domain of u and D' any circle (or ball) with $\mathrm{vol}(D') = \mathrm{vol}(D)$, then

$$\lambda(D) \geq \lambda(D')$$

where $\lambda(D)$ denotes the first eigenvalue of the Laplace-Beltrami operator.

With this method we havev for the two dimensional case (see [CI], p. 93)

$$\limsup_{k\to\infty} \frac{\mu(\psi_k)}{k} \leq \frac{(2\pi/j_0)^2}{\omega_2^2}$$

where j_0 is the first zero of the Bessel function J_0 and ω_2 is the volume of the unit disk in $\mathbb{R}^2$. Since $j_0 \approx 2.40483$ and $\omega_2 = \pi$ we have for spherical harmonics

Theorem 2: (Pleijel's estimate)

$$\limsup_{\ell\to\infty} \frac{\mu(u_\ell)}{\ell(\ell-1)} \leq 0.69$$

where the u_ℓ denote spherical harmonics of degree ℓ.

Numerical experiments make us conjecture, that the maximal number of nodal domains occur in the cases of the base functions Y_ℓ^m. Therefore

Conjecture:

$$\max_{u\in E_\ell} \mu(u) = \begin{cases} \frac{1}{2}(\ell+1)^2 & \text{if } \ell \text{ is odd} \\ \frac{1}{2}\ell(\ell+2) & \text{if } \ell \text{ is even} \end{cases}$$

where E_ℓ denotes the eigenspace to the eigenvalue $\ell(\ell-1)$.

We will show, that this conjecture holds for the case $\ell \leq 6$.

We therefore have for small ℓ from different methods:

Table 1: Estimates for the number of nodal domains, obtained from different methods

ℓ	Courant	Faber-Krahn	Conjecture
3	8	8	8
4	14	14	12
5	22	20	18
6	32	29	24
7	44	38	32

Früher dachte ich, man könnte fortschreiten im Wissen. Man kann das. Aber wohin? Wenn man von einem gegründeten, obzwar nicht ausgeführten Gedanken anfängt, den uns ein anderer hinterlassen, so kann man wohl hoffen, es bei fortgesetztem Nachdenken weiter zu bringen, als der scharfsinnige Mann kam, dem den ersten Funken dieses Lichtes zu verdanken hatte: so Kant über Hume. Der Satz leuchtet ein. Er spricht vom Denken, das wesensmäßig ein Weiterdenken ist. Er spricht vom Fortschreiten im Denken. Was hat das mit dem zu tun, was wir Welt nennen?

Peter Rosei

in: Dumme

Algebraic curves

Real projective plane algebraic curves

In the classical model of the $\mathbb{R}P^2$ all points in $\mathbb{R}^3$ which lie on a straight line through the origin are identified. A real projective plane algebraic curve p is the zero set of a homogeneous polynomial in $(x_0 : x_1 : x_2)$[1]

$$P = \sum_{i+j+k=n} a_{ijk} x_0^i x_1^j x_2^k$$

where $a_{ijk} \in \mathbb{R}$. The degree of the curve p is defined as the degree of the polynomial P.

Remark 1:
For short we will use from now on small letters (e.g. p) to denote algebraic curves and capital letters (e.g. P) to denote the homogeneous polynomial of their equations. If we say *components* of p we always mean the topological components of $p^{-1}(0)$. If we say p decomposes or reduces into several *factors*, then we mean that the polynomial P decomposes into factors of P and each factor of p is the curve determined by a factor of P.
We also will use the symbol p for the "spherical polynomial" $P|_{S^2}$, if there is no risk of confusion.

A good survey of real projective algebraic curves can be found in the the papers of of D. A. Gudkov [GU1] and O. Ya. Viro [VI]. Results and definitions on (complex) projective algebraic curves can be found in the book of R. Walker [WR] and the wonderful text-book of E. Brieskorn and H. Knörrer [BK].

Nodal lines as real algebraic curves

Now if we identify antipodal points of the sphere S^2 we again obtain a model of the real projective plane. Since our spherical harmonics are the restriction of harmonic

[1] The notation $(x_0 : x_1 : x_2) \in \mathbb{R}P^2$ is often used to point out that this are "relative" coordinates of the point. Every multiple of this triple describes the same point.

homogeneous polynomials to the sphere, we can see our nodal lines as *real plane algebraic curves*.

Therefore we can use results concerning real algebraic curves and translate then into results concerning nodal lines of spherical harmonics. This "translation" is done in the following way (see figure 1):

- (•) Each singular point of a real projective plane algebraic curve p is counted as two points of the spherical curve p, i.e. the nodal set of the function $p = P|_{S^2}$.
- (•) Each component of p, which is not fixed under inversion, is counted as two components of the spherical curve. This must be done, since these components always appear as disjoint pairs on the sphere.
- (•) If there is a component, which remains unchanged under inversion, it is counted only once. We call such a component *invariant component*. When this exceptional case occurs, tells us

Lemma 1:

There is at most one invariant component of p. If the degree of p is odd there always exits such a component. If its degree is even and if p is non-singular, then there is none.

Proof:

The first statement is rather simple, since two invariant components must intersect. The other can be found in [VI], §1.3.□

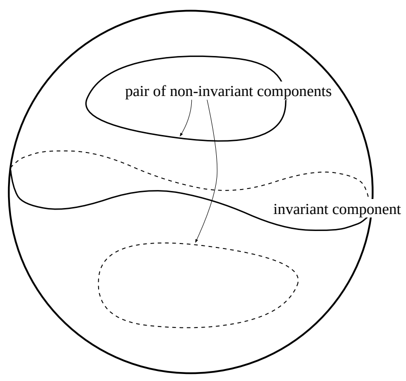


FIGURE 1

Remark 2:

To lessen the risk of confusion we denote by *spherical curve* or simply by *curve* the zero set of a spherical harmonic u or a spherical polynomial p on the sphere S^2, whereas we mean the zero set of the polynomial P on the projective plane $\mathbb{R}P^2$ when we say *projective curve*.

Remark 3:

For later uses we define the *inside* of an oval (or in general of a simply closed arc) to be the smaller of the two sets in which the oval (curve) divides the sphere. The *outside* is the other set (i.e. the interior of the complement of the inside). If both sets are equal — that is the curve remains invariant under inversion — there is no inside.

Some remarks on algebraic geometry:

Classification problems are one of the most interesting topics in algebraic geometry. So a lot of methods have been developed for investigating the topology of singular points of (complex) projective algebraic curves (see e.g. [BK], §8.).

In the field of real algebraic geometry a lot of work is done on the study of the topology of real projective algebraic curves (*Hilbert's sixteenth problem*). But this is a very hard job to do. The classification of real non-singular projective curves of degree less than or equal to five is trivial. Whereas it has last from 1900, when D. Hilbert formulated this problem first, till 1969, when D. A. Gudkov made a complete classification, to solve this problem for non-singular projective curves of degree 6 (all possible 56 isotopy types[2] were listed in [VI], §1.13B). For curves of higher degree just partial solutions have been made (see for example the work of O. Ya. Viro).
For the case of real singular projective curves Gudkov et al. ([GU3]) made a complete classification of curves of degree 4. They have found 99 possible isotopy types. For the cases of degree five or higher I could not find any result.

Another field of investigation have been (so called) M-curves and their constructions. These are projective curves which have exactly $g+1$ complete real branches, where

[2] Two topological spaces X and Y have the same isotopy type, if there exist two maps $f : X \to Y$ and $g : Y \to X$, such that $f \circ g$ and $g \circ f$ are firstly homotop to the identity and secondly these homotopy maps are homeomorphisms.

g is the genus of the curve. A. Harnack not only proved in [HA] that the number of branches of any projective curve does not exceed $g+1$, he constructed such M-curves. There were a lot of results on M-curves and (M$-$1)-curves (see [VI], §1.13 and §2.). Several methods have been found by Harnack, Hilbert, Viro and others to construct M-curves of different isotopy type.

We already know from Bers' theorem, that the (real) topology of singular points of spherical harmonics is very simple. We will see later, that M-curves cannot be harmonic. So most of the methods and results on algebraic curves are not useful for the investigations of spherical harmonics. Thus we should not promise good results from these methods.

Harnack's theorem

Lemma 2: (HARNACK'S THEOREM)
Let p be an irreducible real projective plane algebraic curve. Then the number of components of p does not exceed $g+1$, where g is the genus of the curve p.

Lemma 3: (ABOUT THE GENUS OF A CURVE)
Let p be an irreducible real projective plane algebraic curve of degree n. Then the *genus* is given by

$$g = \frac{(n-1)(n-2)}{2} - \sum \frac{\tilde{\varrho}_p(x)\,(\tilde{\varrho}_p(x)-1)}{2}$$

where the sum is taken over all singular points and $\tilde{\varrho}_p(x)$ is the multiplicity of the singular point x of the projective curve p.

A proof [3] of this result can be found in [BK], §9.2, p. 804. An outline of the proof of Harnack's theorem is given later. There the genus of an algebraic curve is roughly explained, too.

From this we easily obtain the following corollary.

[3] Although this result is "well known" ([GU1],[WI]), it is very hard to find even the result itself. It can be found in [EE], p. 373 or in [BK], p. 820 for example.

Lemma 4:
Let p be an irreducible spherical polynomial of degree ℓ. Then

$$\zeta(p) + \nu(p) \leq (\ell - 1)(\ell - 2) + \kappa_\ell$$

where

$$\kappa_\ell = \begin{cases} 2 & \text{if } \ell \text{ is even} \\ 1 & \text{if } \ell \text{ is odd} \end{cases}$$

Proof:
Suppose ℓ is odd. Then by lemma 1 and Harnack's theorem the number of components of p does not exceed $2g + 1$, i.e. $\zeta(p) \leq 2g + 1$. Since every singular point of the projective curve p is counted as two singular points of the spherical curve p and $\tilde{\varrho}_p(x) \geq 2$ for every singular points we have

$$\nu(p) = \sum (\varrho_p(x) - 1) = 2 \sum (\tilde{\varrho}_p(x) - 1) \leq 2 \sum \frac{\tilde{\varrho}_p(x)(\tilde{\varrho}_p(x) - 1)}{2}$$

Hence $\zeta(u) + \mu(u) \leq 2g + 1 + 2 \sum \frac{\tilde{\varrho}_u(x)(\tilde{\varrho}_u(x) - 1)}{2}$. By inserting the result of lemma 3 we obtain the desired estimate.
The case for even ℓ is the same. But then there might not be an invariant component of p.□

From Euler's theorem we immediately obtain

Theorem 1:
Let p be an irreducible spherical curve of degree ℓ. Then

$$\mu(p) \leq (\ell - 1)(\ell - 2) + \kappa_\ell + 1$$

Bezout's theorem

For the investigations on reducible curves we need

Lemma 5: (Bezout's theorem)
If two projective curves of degree m and n have no common factors, they have at most mn common points.

This is the weak form of Bezout's theorem. It can be shown that the projective curves have exactly mn common points in the complex projective plane. But since we are interested in the real points only, lemma 5 suffices. For the proof we need the concept of the resultant of two polynomials.

Definition: Let

$$F_n = A_n + A_{n-1}x_2 + \ldots + A_0 x_2^n$$
$$G_m = B_m + B_{m-1}x_2 + \ldots + B_0 x_2^m$$

where A_i, B_i are homogeneous polynomials of degree i in x_0, x_1. Then

$$R(x_0, x_1) = \begin{vmatrix} A_n & A_{n-1} & \ldots & \ldots & A_0 & & & & \\ & A_n & A_{n-1} & \ldots & \ldots & A_0 & & & \\ & & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & & A_n & A_{n-1} & \ldots & \ldots & A_0 \\ B_m & B_{m-1} & \ldots & \ldots & B_0 & & & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & & B_m & B_{m-1} & \ldots & \ldots & B_0 \end{vmatrix}$$

is called the *resultant* of F and G with respect to x_2.

In the following lemma we summarize the properties of the resultant.

Lemma 6:

Let F, G and R be defined as above. Then

(a) $R(x_0, x_1)$ is either a homogeneous polynomial of degree mn or equal to zero.

(b) $R(c_0, c_1) = 0$ if and only if there is a c_2 so that $(c_0 : c_1 : c_2)$ is a common point of the curves f and g.

(c) $R(x_0, x_1) \equiv 0$ if and only if f and g have a common factor.

The proofs of these results can be found in [WR] ((a) and (c) are equivalent to theorem 10.9 and theorem 10.7, respectively, (b) immediately follows from theorem 9.3.).

Proof of lemma 5: ([WR], p. 59)

Suppose f and g have more than mn common points. Select any $mn+1$ of these and join each pair of selected points by a line. Since there is only a finite number of such lines there is a point y not on any of the lines nor on f or g. Choose a coordinate system in which $y = (0, 0, 1)$ and calculate the resultant of F and G with respect to x_2. By lemma 6(b) the first two coordinates of any point common to f and g satisfy $R(x_0, x_1) = 0$. But each of the $mn + 1$ selected points has a different value of $c_0 : c_1$, since no pair of them is collinear with $y = (0, 0, 1)$. Hence by lemma 6(a) $R(x_0, x_1)$ must be zero (Otherwise it would be a homogeneous polynomial of degree mn in $x_0 : x_1$ with at least $mn + 1$ zeros, a contradiction.) and by lemma 6(c) f and g must have a common factor.□

For later uses we need the following slight improvement of lemma 5, which can be shown by a little modification of the above proof (see [WR], p. 61).

Lemma 5a: (Bezout's theorem)
If two spherical curves p and q of degree m and n have no common factors, then

$$\sum \varrho_p(x) \cdot \varrho_q(x) \leq 2mn$$

where the sum is taken over all common points of p and q.

Proof of Harnack's theorem

Harnack's theorem can be proved in several ways. A. Harnack himself gave in [HA] a proof for non-singular curves and curves with at most simple double points by using Bezout's theorem. He constructed a second curve of degree $n - 2$ which contains each double point and one point on each "oval" of p. Then he used the fact that each oval of the curve must be cut twice by any other curve. The result follows from Bezout's theorem. For details see [HA] or [GU1], §3.
This method is difficult and due to Gudkov ([GU1], §3.) it has not been carried through for the cases when the singularities of p are arbitrary. Nevertheless it can be used to prove the weaker statement of lemma 3 directly.

Another proof is according to F. Klein ([KF1], p. 124). It uses the fact that g is the genus of the Riemann surface of $\mathbb{C}p$. We give a short outline of his proof for non-singular curves only in a version which can be found more detailed in [WI] or [GU1], §3.
Let $\mathbb{C}p$ be the "complexification" of p, i.e. all points in the complex projective plane $\mathbb{CP}^2$ which satisfy $P = 0$. Then $p \subset \mathbb{C}p$. p is the set of all fixed point under the

complex conjugation T. $\mathbb{C}p \backslash p$ has at most two components which interchange by T. If we remove all but one component of p from $\mathbb{C}p$, the resulting space is connected via the remaining component of p.
$\mathbb{C}p$ is an orientable compact connected surface. The genus g then gives the number of handles of $\mathbb{C}p$. It is easy to see, that we can remove at most g circles unless the surface disconnects. Hence p has at most $g+1$ components.

If p is singular we obtain by disolution of the singular points a curve p' without singularities. Then the genus of p is defined as the genus of p' (see [BK], p. 814 ff) and the result follows from the non-singular case.

The idea of F. Klein enlightens the connection between the genus[4] of the curve and its maximal number of components. The method of A. Harnack on the other hand gives us hints how to continue our investigations.

Reducible curves

Now we want to use Harnack's and Bezout's theorem to deduce an upper bound for the number of "nodal domains" for arbitrary homogeneous polynomials. Obviously we then have such an estimate for the special case of harmonic homogeneous polynomials and hence for the spherical harmonics.

Theorem 2:
Let p be a spherical polynomial of degree ℓ. Then

$$\mu(p) \leq \ell(\ell-1)+2$$

Equality holds if and only if p consists of ℓ straight lines (linear factors), where not more than two of them intersect in one point.

Remark 4:
This is exactly the same bound we obtain from Courant's nodal domain theorem! V. I. Arnol'd suggested in [AR1] the following method to proof theorem 2. He used the fact, that every homogeneous polynomial of degree ℓ can be expressed in terms of

[4] In some books about algebraic geometry the genus of a curve is defined in a completely algebraic way as a rather mysterious invariant. They do not even mention this geometric meaning of the genus. See e.g. [WR] or [VW3].

harmonic homogeneous polynomials of degree less than or equal to ℓ. Then he uses the theorem of Courant-Hermann (see [CH], p. 394, footnote 2), a generalization of Courant's theorem. Arnol'd noticed that this theorem unluckily is *false (!)*. But according to him to prove theorem 2 it suffices to prove the Courant-Hermann theorem for the sphere with the standard metric (see also [GU1], §9.2).

But this does not enlighten this curious connection between Courant's analytic result for harmonic functions and Harnack's algebraic result for homogeneous polynomials. The fact that the Courant-Hermann theorem is true for this special case (otherwise theorem 2 would be false), does not explain this. Later we will find a similar connection when we investigate non-singular spherical harmonics by analytic methods. Then we will derive an estimate which is exactly like that of Harnack's theorem.

The same curious result can be found for the solutions of the two dimensional harmonic oscillator, i.e. of $\left(-\Delta + |x|^2\right)\psi(x) = (2n+1)\psi(x)$ in $\mathbb{R}^2$, where $n \in \mathbb{N}_0$. The solutions of this system are functions of the type $P(x_1, x_2) \cdot e^{-\frac{|x|^2}{2}}$, where P is a (so called *Hermite*-) polynomial in x_1, x_2 of degree n (see e.g. [AG]). When we calculate the number of their nodal domains, we again can either use Courant's theorem or Harnack's theorem. Again we arrive in in both cases at the same bound. Even if we use the symmetry of the Hermite polynomials this remains true.

Now back to theorem 2. Suppose the spherical harmonics u consists of ℓ straight lines. Then if exactly two of them intersect in one point they must be perpendicular by Bers' theorem. If u has the maximum number of nodal domains all lines must be perpendicular to each other by theorem 2, which is not possible if $\ell > 3$. Hence

Theorem 3:
Let u be a spherical harmonic of degree $\ell \geq 4$, then

$$\mu(u) \leq \ell(\ell-1)$$

For the case $\ell = 4$ we obtain $\mu(u_4) \leq 12$, a sharp bound according to lemma 1 about spherical harmonics.

Now to the proof of theorem 2.

Lemma 7:

Let p_1, p_2 be two spherical curves with a common point x. Then

$$\varrho_{p_1 \cdot p_2}(x) = \varrho_{p_1}(x) + \varrho_{p_2}(x)$$

Proof:

The multiplicity of p_i in x is the order of the lowest non-vanishing term of its Taylor expansion ([BK], p. 271). From this the result follows.□

Lemma 8: (about the union of two curves)

Let p_m, p_n be two spherical curves of degree m and n without a common component. Then

$$\nu(p_m \cdot p_n) \leq \nu(p_m) + \nu(p_n) + 2mn \tag{1}$$

$$\zeta(p_m \cdot p_n) \leq \zeta(p_m) + \zeta(p_n) \tag{2}$$

The first equality holds if and only if p_m and p_n have exactly mn common points and with $\varrho_{p_m}(x) = \varrho_{p_n}(x) = 1$ at each of these, the second if and only if p_m and p_n do not intersect.

Proof:

Let S_m and S_n be the set of singular points of p_m and p_n, S_c the set of common points of p_m and p_n. Then $S = S_c \cup S_m \cup S_n$ contains all singular points of $p_m \cdot p_n$, and by lemma 7

$$\begin{aligned}
\nu(p_m \cdot p_n) &= \sum_{S_m \setminus S_c} \varrho_{p_m}(x) + \sum_{S_n \setminus S_c} \varrho_{p_n}(x) + \sum_{S_c} (\varrho_{p_m}(x) + \varrho_{p_n}(x)) - |S| \\
&= \underbrace{\sum_{S_m \setminus S_c} (\varrho_{p_m}(x) - 1)}_{\leq \nu(p_m)} + \underbrace{\sum_{S_n \setminus S_c} (\varrho_{p_n}(x) - 1)}_{\leq \nu(p_n)} + \sum_{S_c} (\varrho_{p_m}(x) + \varrho_{p_n}(x) - 1)
\end{aligned}$$

By Bezout's theorem we have

$$\sum_{S_c} (\varrho_{p_m}(x) + \varrho_{p_n}(x) - 1) \leq \sum_{S_c} \varrho_{p_m}(x) \varrho_{p_n}(x) \leq 2mn$$

and the inequality follows. All the equalities hold if and only if $S_m \setminus S_c = S_m$ and $S_n \setminus S_c = S_n$, i.e. if no common point of p_m and p_n is a point of multiplicity greater than 1 of p_m or p_n, as claimed.

Inequality (2) is trivial.□

Proof of theorem 2:

From theorem 1 we know that theorem 2 is true for irreducible curves. Moreover we obtain, that a irreducible curve has the maximum number of "nodal domains" if $\ell = 1$, i.e. if it is a single straight line.

Now let $p = p_m \cdot p_{\ell-m}$ be reducible with factors p_m and $p_{\ell-m}$ of degree m and $\ell - m$. Without loss we can remove all multiple factors of p, since the number of "nodal domains" remains unchanged. By induction we can assume

$$\nu(p_m) + \zeta(p_m) \leq m(m-1) + 1$$

$$\nu(p_{\ell-m}) + \zeta(p_{\ell-m}) \leq (\ell - m)(\ell - m - 1) + 1$$

Now we can use lemma 8 and note that equality (1) and (2) cannot hold at the same time.

$$\begin{aligned}
\nu(p) + \zeta(p) &< (\nu(p_m) + \zeta(p_m)) + (\nu(p_{\ell-m}) + \zeta(p_{\ell-m})) + 2m(\ell - m) \\
&\leq m(m-1) + 1 + (\ell - m)(\ell - m - 1) + 1 + 2m(\ell - m) \\
&= \ell^2 - \ell + 2
\end{aligned}$$

and so

$$\nu(p) + \zeta(p) \leq \ell(\ell - 1) + 1$$

Together with Euler's theorem this is the desired result.

The maximum number of nodal domains occurs if and only if $\zeta(p_m) + \nu(p_m)$ and $\zeta(p_{\ell-m}) + \nu(p_{\ell-m})$ are maximal and no common point of p_m and $p_{\ell-m}$ is a singular point of p_m or $p_{\ell-m}$ (lemma 8). By induction this is only possible if p consists of ℓ straight lines, where not more than two of them intersect in one point.□

In [KA2] V. N. Karpushkin tried to improve theorem 3. He also used the theorems of Harnack, Bezout and Bers. After a short calculation he drived an estimate from which the following can be deduced

Theorem 4:

Let u be a spherical harmonic. Then

$$\mu(u) \leq \ell(\ell - 2) + 5$$

Although this is better than theorem 3 in general, neither it gives a sharp bound for small ℓ (e.g. $\mu(u_4) \leq 13$) nor it is asymptotically better than Courant's theorem.

A short review of the proof of theorem 2 gives us evidence, that we cannot obtain a better bound than those for the irreducible factors of the harmonic polynomials. But at the time Harnack's theorem is the best thing we have. Since Harnack's theorem asymptotically is as bad as Courant's theorem, we can use this method just for finding sharp bounds for small ℓ. Better results for greater ℓ cannot be found unless the irreducible factors of harmonic homogeneous polynomials are better understood. Karpushkin's work moreover shows us, that we need further studies of the location of irreducible factors. Otherwise we would not get better results.

It would be a nice thing to have a birational transform on projective curves — like the quadratic (or Cremona) transform — which maps harmonic curves (defined in an algebraic way) into harmonic curves with "nice" properities. Since birational transforms does not change the genus of a curve (see [VW3], p. 99ff), we could save a lot of time. Since we do not know such a transform, we have to consider a lot of different cases.

Der Ochse sagte eines Tages zum Schwan: Leih mir deine Flügel, so leihe ich dir meine Hörner! — Neugierig, wie er war, ging der Schwan auf das Anerbieten ein. Abends trafen die beiden wieder zusammen.
Was hast du erlebt, fragte der Ochse den Schwan.
Gar nichts, sagte der enttäuscht, niemand hatte Respekt vor mir; ich bin ausgelacht worden.
Und wie war es bei dir, fragte er den Ochsen. Herrlich, sagte der Ochs, für gewöhnlich stören mich die Fliegen beim Fressen; heute fraß ich ungestört.

Peter Rosei
Ochse und Schwan

Straight lines

In this chapter we want to deduce properties of spherical harmonics which decomposes into one or more linear factors. We will need this properties later, when we want to show a sharp bound for the case of degree 6. But first we make a complete classification of those spherical harmonics, which consist of straight lines only, and state a nice theorem.

Platonic solids

Theorem 1: (Spherical harmonics and Platonic solids)
Let u be a spherical harmonic, which completely decomposes into linear factors. Then the nodal set of u is an union of ℓ different straight lines which either satisfy one of the two properties

(a) All lines intersect in one point, i.e. u possesses one singular point of multiplicity ℓ.

(b) u possesses one singular point of multiplicity $(\ell-1)$ at the pole of the sphere and $2(\ell-1)$ singular points of multiplicity 2 situated at the equator.

or is one of the exceptions

(c) $\ell=6$ and u possesses 4 singular points of multiplicity 3 and 3 singular points of multiplicity 2.

(d) $\ell=9$ and u possesses 3 singular points of multiplicity 4, 4 singular points of multiplicity 3 and 6 singular points of multiplicity 2.

(e) $\ell=15$ and u possesses 6 singular points of multiplicity 5, 10 singular points of multiplicity 3 and 15 singular points of multiplicity 2.

Moreover for each of the cases (a) – (e) there exists a spherical harmonic with this property (see figure 1) and these are unique to within rotation for (c), (d) and (e).

Remark:
As can be seen from the proof of theorem 1, these cases are closely related to the finite groups of revolution (finite subgroups of $SO(3)$) and hence to the regular polyhedra (*"Platonic solids"*) (see e.g. [BE1], p. 22 ff or [CX], §15.4, p. 329 ff).

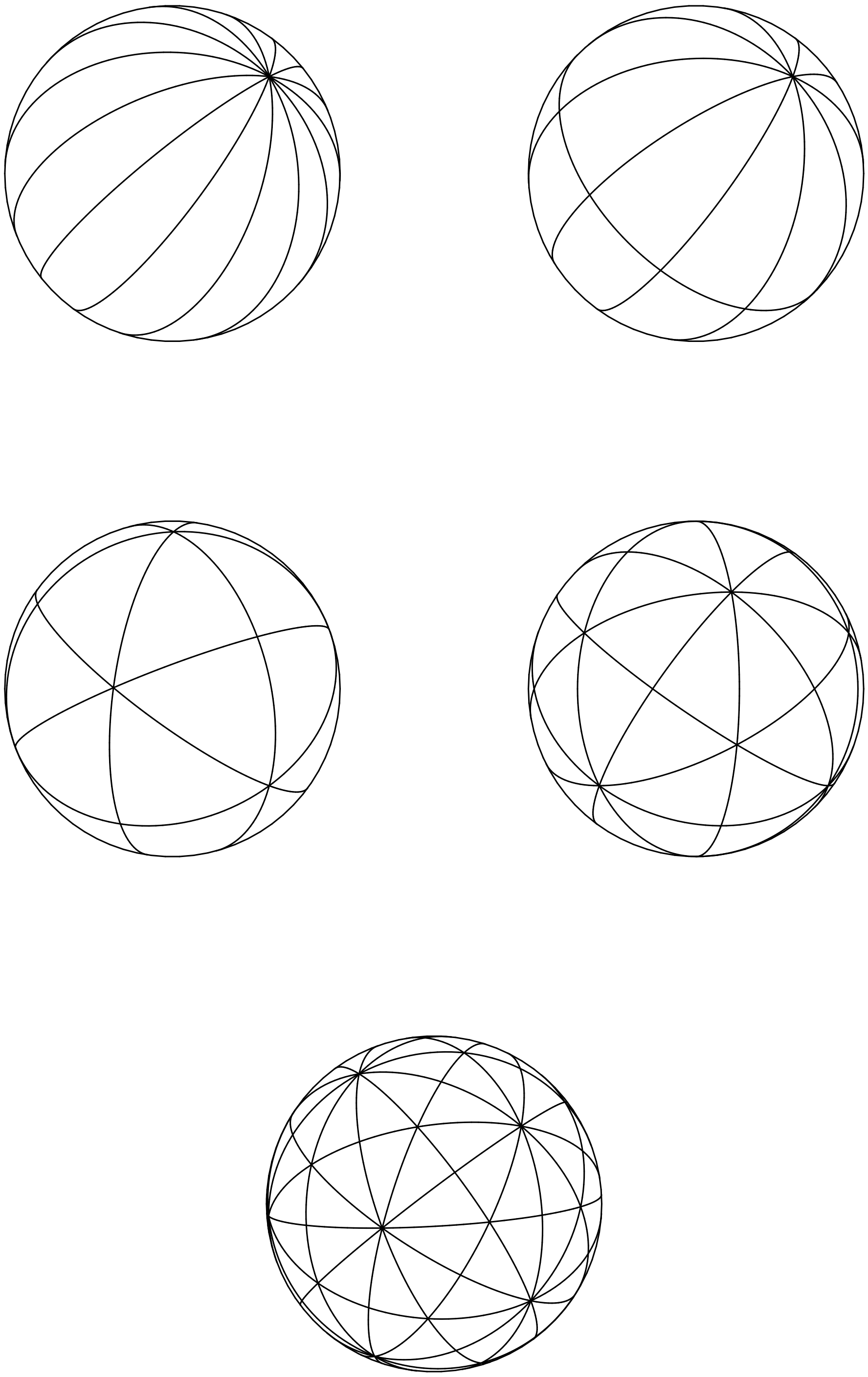


Figure 1: Spherical harmonics of types (a) – (e)

Proof of theorem 1:

Suppose u does neither fulfill case (a) nor (b). Then the nodal domains of u are spherical polygons of n sides with angles $\frac{\pi}{k_1}, \ldots, \frac{\pi}{k_n}$, $k_i \in \mathbb{N}$ (Bers's theorem) and $n \geq 3$. Now the sum of these angles is

$$\sum_{i=1}^{n} \frac{\pi}{k_i} \leq \tfrac{n}{2}\pi$$

since $k_i \geq 2$. On the other hand we must have for such polygons

$$\sum_{i=1}^{n} \frac{\pi}{k_i} > (n-2)\pi$$

Hence $n \leq 3$, and all nodal domains are triangles with singular points of u of multiplicity k_i as vertices. Such a vertex is therefore the corner of $2k_i$ triangles. By congruence theorem all these triangles must be equal and have angles $\frac{\pi}{k}$, $\frac{\pi}{l}$ and $\frac{\pi}{m}$ which satisfy

$$\left(\tfrac{1}{k} + \tfrac{1}{l} + \tfrac{1}{m}\right)\pi > \pi$$

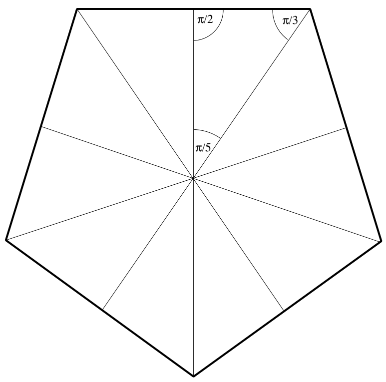


Figure 2: The triangles of case $m = 5$ form a pentagon

If two angles are equal to $\frac{\pi}{2}$ the union of two of the triangles is a sector and hence u has property (b).

So the following cases remain:

$$k = 2, \quad l = 3, \quad m \in \{3, 4, 5\}$$

For each case there is a singular point of multiplicity m. All triangles, which have such a point as vertex, form a regular spherical polygon with m sides. All these regular polygons are congruent. Their vertices are singular points of multiplicity 3, the centers of their edges singular points of multiplicity 2 (see figure 2).

Furthermore these polygons are a decomposition of the sphere and each vertex is vertex of at most 3 polygons (otherwise it would be a singular point of u of multiplicity greater than 3). All these regular polygons can be constructed when we inscribe a regular polyhedra (a "Platonic solid") into the unit-ball and project its edges to the sphere. For this we can use only those polyhedra, where each vertex is in the intersection of at most 3 faces, which are tetrahedron ($m = 3$), cube ($m = 4$) and dodecahedron ($m = 5$) (see e.g. [BE1], p. 25). The number of singular points of u of multiplicity k, l and m result from the number of edges, vertices and faces, respectively (keep in mind, that a pair of antipodal points is counted as *one* point.). The degree of u results from the number of lines which are necessary to split the sphere into the nodal domains of u (i.e. the triangles).

Now we show, that there exists a spherical harmonic for each case. Obviously $Y_\ell^\ell(\theta,\varphi)$ and $Y_\ell^{\ell-1}(\theta,\varphi)$ are of type (a) and (b), respectively.

Now we construct by means of tetrahedron, cube and icosahedron (since the icosahedron group is isomorphic to the dodecahedron group) linear factors for homogeneous polynomials in $\mathbb{R}^3$, $P_\ell(x) = \prod_{i=1}^{\ell}\langle a_i^\ell, x\rangle$, which satisfy the cases (c), (d) and (e), respectively. We have for the a_i^ℓ:

$$\begin{array}{lll}
a_1^6 = (-1,0,0) & a_2^6 = (1,\sqrt{3},0) & a_3^6 = (1,-\sqrt{3},0) \\
a_4^6 = (-1,\frac{\sqrt{3}}{3},2\frac{\sqrt{2}}{\sqrt{3}}) & a_5^6 = (-1,-\frac{\sqrt{3}}{3},-2\frac{\sqrt{2}}{\sqrt{3}}) & a_6^6 = (0,-2\frac{\sqrt{3}}{3},2\frac{\sqrt{2}}{\sqrt{3}})
\end{array}$$

$$\begin{array}{lll}
a_1^9 = (1,0,0) & a_4^9 = (1,1,0) & a_7^9 = (-1,0,1) \\
a_2^9 = (0,1,0) & a_5^9 = (-1,1,0) & a_8^9 = (0,1,1) \\
a_3^9 = (0,0,1) & a_6^9 = (1,0,1) & a_9^9 = (0,-1,1)
\end{array}$$

and

$$\begin{array}{ll}
a_1^{15} = (0,1,0) & \\
a_2^{15} = (-\sin(\frac{2\pi}{5}),\cos(\frac{2\pi}{5}),0) & a_4^{15} = (-\sin(\frac{6\pi}{5}),\cos(\frac{6\pi}{5}),0) \\
a_3^{15} = (-\sin(\frac{4\pi}{5}),\cos(\frac{4\pi}{5}),0) & a_5^{15} = (-\sin(\frac{8\pi}{5}),\cos(\frac{8\pi}{5}),0)
\end{array}$$

$$a_6^{15} = (-\sin(\tfrac{2\pi}{5}), \cos(\tfrac{2\pi}{5}) - 1, \alpha\sin(\tfrac{2\pi}{5}))$$
$$a_7^{15} = (-\sin(\tfrac{4\pi}{5}), \cos(\tfrac{4\pi}{5}) - 1, \alpha\sin(\tfrac{4\pi}{5}))$$
$$a_8^{15} = (-\sin(\tfrac{6\pi}{5}), \cos(\tfrac{6\pi}{5}) - 1, \alpha\sin(\tfrac{6\pi}{5}))$$
$$a_9^{15} = (-\sin(\tfrac{6\pi}{5}), \cos(\tfrac{8\pi}{5}) - 1, \alpha\sin(\tfrac{8\pi}{5}))$$
$$a_{10}^{15} = (\sin(\tfrac{2\pi}{5}) - \sin(\tfrac{4\pi}{5}), \cos(\tfrac{2\pi}{5}) - \cos(\tfrac{4\pi}{5}), \alpha\sin(\tfrac{2\pi}{5}))$$
$$a_{11}^{15} = (\sin(\tfrac{2\pi}{5}) - \sin(\tfrac{6\pi}{5}), \cos(\tfrac{2\pi}{5}) - \cos(\tfrac{6\pi}{5}), \alpha\sin(\tfrac{4\pi}{5}))$$
$$a_{12}^{15} = (\sin(\tfrac{2\pi}{5}) - \sin(\tfrac{8\pi}{5}), \cos(\tfrac{2\pi}{5}) - \cos(\tfrac{8\pi}{5}), \alpha\sin(\tfrac{6\pi}{5}))$$
$$a_{13}^{15} = (\sin(\tfrac{4\pi}{5}) - \sin(\tfrac{6\pi}{5}), \cos(\tfrac{4\pi}{5}) - \cos(\tfrac{6\pi}{5}), \alpha\sin(\tfrac{2\pi}{5}))$$
$$a_{14}^{15} = (\sin(\tfrac{4\pi}{5}) - \sin(\tfrac{8\pi}{5}), \cos(\tfrac{4\pi}{5}) - \cos(\tfrac{8\pi}{5}), \alpha\sin(\tfrac{4\pi}{5}))$$
$$a_{15}^{15} = (\sin(\tfrac{6\pi}{5}) - \sin(\tfrac{8\pi}{5}), \cos(\tfrac{6\pi}{5}) - \cos(\tfrac{8\pi}{5}), \alpha\sin(\tfrac{2\pi}{5}))$$

where

$$\alpha = \frac{1}{\sqrt{\sin(\frac{\pi}{5})^2 - \sin(\frac{\pi}{10})^2}}$$

A very tedious calculation shows, that these polynomials are harmonic and hence their restrictions p to the sphere are spherical harmonics. Since they are not of type (a) or (b), they must be of type (c), (d) and (e), respectively.$\square$

Since polynomials are uniquely determined (except for a multiple) by their nodal set we have an interesting corollary.

Lemma 1:
Every product of linear functions which fulfills the Bers property is harmonic.

Linear factors

Now we want to make some investigations on spherical harmonics which decomposes into several linear terms. We will show

Theorem 2:
The product u_{lin} of all linear factors of a spherical harmonic u is a spherical harmonic.

For verifying this we want to represent u in the following way:

$$u(\theta, \varphi) = \sum_{i=0}^{\ell} u^i(\theta, \varphi)$$

where

$$\begin{aligned} u^0(\theta,\varphi) &= \alpha_0 Y_\ell^0(\theta,\varphi) = \alpha_0 c_{\ell m} P_\ell(\cos\theta) \\ u^i(\theta,\varphi) &= \alpha_i Y_\ell^i(\theta,\varphi) + \alpha_{-i} Y_\ell^{-i}(\theta,\varphi) \\ &= (\alpha_i \sin i\varphi + \alpha_{-i}\cos i\varphi) c_{\ell i} P_\ell^i(\cos\theta) \end{aligned}$$

Lemma 2:

Let u be a spherical harmonic and L a linear factor of u which

(a) either contains the pole

(b) or equals the equator.

Then all functions u^i of u either are divisible by L or are equal to 0.

Proof:

(a) Since L contains the pole, $\mathcal{N}(L) = \{(\theta,\varphi) : \varphi = \varphi_L \vee \varphi = \varphi_L + \pi\}$. Let $u_i'(\theta) = u^i(\theta,\varphi_L) = d_i P_\ell^i(\cos\theta)$. By assumption $u|_L \equiv 0$. Hence

$$0 = \sum_{i=0}^{\ell} u_i' = \sum_{i=0}^{\ell} d_i P_\ell^i(\cos\theta)$$

Since the Legendre polynomials $P_\ell^i(\cos\theta)$, $i = 0,\dots\ell$, are linearly independent (see e.g. [AG], p. 562), $d_i = 0$ for all i. Hence either $u_i \equiv 0$ or is divisible by L.

(b) can be shown analogously to (a) by using the linear independency of the trigonometric functions.□

Lemma 3:

Suppose u contains k linear factors ($k \geq 1$) which intersect in the pole, then

$$u(\theta,\varphi) = \sum_{i=1}^{[\frac{\ell}{k}]} u^{i\cdot k}(\theta,\varphi)$$

Proof:

All functions u^i, which do not vanish, are divisible by all these linear factors (lemma 2). But since the nodal lines of u^i consist of i meridians, which form an equiangular system at the pole (see lemma 1 about spherical harmonics), all u^i vanish if i is not an integral multiple of k.□

Now we easily can prove theorem 2.

Proof of theorem 2:
Since every polynomial of degree less than 2 is harmonic, we can assume that u decomposes in at least two linear factors. Suppose x_1 is the intersection point of k factors. We can choose x_1 as pole of our coordinate system. Then by the properties of the functions u^i and lemma 3 these factors form an equiangular system at x_1. Since this is true for all such intersection point of linear factors, u_{lin} must be a spherical harmonic by lemma 1.□

Special cases

Let $k(u)$ denote the maximal number of linear factors of u which intersect in one point. We want to calculate the number of some special spherical harmonics in terms of $k(u)$.

Lemma 4:
If $k(u) > \frac{\ell}{2}$, then $\mu(u) = 2k(u)(\ell - k(u) + 1)$.

Proof:
We choose the intersection point of the $k(u)$ lines as pole of our coordinate system. From lemma 3 we have $u(\theta, \varphi) = u^{k(u)}(\theta, \varphi)$. From this the result follows.□

Lemma 5:
If $k(u) > \frac{\ell}{3}$, then $\mu(u) \leq 2k(u)(\ell - k(u) + 1)$.

Proof:
Again we have from lemma 3

$$u(\theta, \varphi) = u^{k(u)}(\theta, \varphi) + u^{2k(u)}(\theta, \varphi)$$

We furthermore can assume that one of the linear factors is the zero meridian (i.e. $\varphi = 0$). Then all spherical harmonics u, which fulfill the assumptions of the lemma, can be written as (a multiple of)

$$u_\lambda(\theta, \varphi) = (1 - |\lambda|) \sin k(u)\varphi \cdot P_\ell^k(\cos\theta) + \lambda \sin 2k(u)\varphi \cdot P_\ell^{2k(u)}(\cos\theta)$$

where $\lambda \in [-1, 1]$. The case $|\lambda| = 1$ is already shown in lemma 4.

Now let S be a sector with angle $\frac{\pi}{k(u)}$, the boundary of which is formed by some linear factors of u. Then each nodal domain, which cuts S, must already be a subset of S.

If $\lambda = 0$, then S will be split by the latitude circles of $u_0 = u^{k(u)}$ into $(\ell - k(u) + 1)$ "quadrangles" V_j, where the "quadrangles" V_0 and $V_{\ell - k(u)+1)}$ contain the pole as "degenerated edge". The quadrangles are numbered, so that V_i and V_{i+1} are always neighboring (see figure 3).

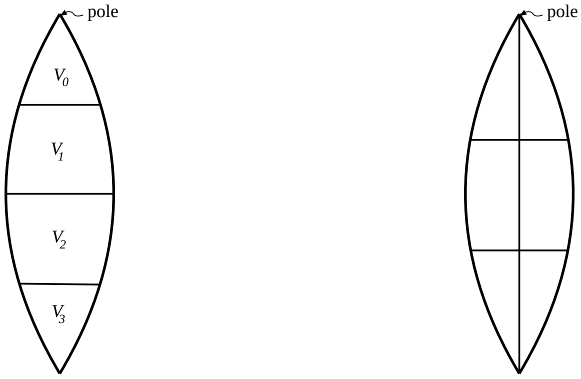


Figure 3: The nodal domains of $u^{k(u)}$ and $u^{2k(u)}$ inside the sector S.

By variation principle any quadrangle V_j must not contain a nodal domain of u_λ. Hence each nodal domain D in S intersects some quadrangles $V_{j_1}, V_{j_1+1}, \ldots, V_{j_2}$ where $j_2 > j_1$. Now we count the number of nodal domains in each quadrangle in the following way: The nodal domain D is counted in V_{j_1} and V_{j_2} (i.e. those quadrangles, where D intersects one edge only) as $\frac{1}{2}$ nodal domain, whereas in $V_{j_1+1}, \ldots, V_{j_2-1}$ (where D intersects two edges) we do not count it. As can easily be seen, altogether a nodal domains counts one nodal domain if we sum over all quadrangles. Hence the sum of these numbers over all quadrangles is the number of nodal domains of u in the sector S.

Since $u^{k(u)}$ vanishes at the boundary of V_j, we have $\mathcal{N}(u_\lambda) \cap \partial V_j = \mathcal{N}(u^{2k(u)}) \cap \partial V_j$ for all $\lambda \neq 0$. The nodal set of $u^{2k(u)}$ in S consists of latitude circles and one meridian in the interior of S (see figure 3 right hand side). Hence each edge of a quadrangle intersects either 0 or 2 nodal domains of u_λ.

Now let $\lambda \in [0, 1)$. Since $u^{k(u)}$ does not change sign in V_j, all the (e.g.) positive nodal domains of u_{λ_1} are contained in the positive nodal domains of u_{λ_2} if $\lambda_1 \leq \lambda_2$. Now either $u^{2k(u)}$ has in V_j one positive nodal domain, which intersects two edges of the quadrangle (see figure 4 left hand side). Or this is true for u_λ, for any $\lambda < 1$ (see figure 4 right hand side). So we can count (at most) one negative and no positive nodal domain in each quadrangle (or vice versa) (see figure 4, black domains). Since the same is true for $\lambda \in (-1, 0]$, each sector has at most $(\ell - k(u) + 1)$ nodal domains. Since there are $2\,k(u)$ of these sectors the result follows.□

Figure 4: The two cases how the nodal domains of $u^{2k(u)}$ intersect the "quadrangle" V_j

It is obvious that we cannot continue in this way. Already the next case, where we assume $k(u) > \frac{\ell}{4}$ becomes to complicated. So we will show another type of result, which we will use later.

Another symmetry

Lemma 6:

Let L be a linear factor of the spherical harmonic u. Then u is anti-symmetric with respect to L.

Proof:

By a proper choice of the coordinate system, L is the zero meridian (i.e. $\varphi_L = 0$). Then by lemma 2

$$u(\theta, \varphi) = \sum_{i=1}^{\ell} \alpha_i P_\ell^i(\cos\theta) \sin i\varphi \tag{1}$$

Now let $x \in S^2$ be any point not on L with spherical coordinates (θ_x, φ_x). Then x' with coordinates $(\theta'_x, \varphi'_x) = (\theta_x, 2\pi - \varphi_x)$ is the point which is symmetric to x with respect to L. Inserting this into (1) we yield $u(x') = -u(x)$ as proposed.□

Lemma 7:

Let x be perpendicular to L, i.e. x is the center of the hemisphere cut out by L. Then u is symmetric with respect to x if ℓ is odd, and anti-symmetric if ℓ is even.

Proof:

Let $y \in S^2$ be any point not on L and y' the point symmetric to y with respect to x. By reflecting y on the line L and taking its antipodal we can map y into y'. Hence from lemma 6 and inversion symmetry of u we have $u(y') = (-1)^\ell(-u(y))$ as proposed.□

Lemma 8:

Let u be a spherical harmonic, which decomposes into several linear factors, i.e. $k(u) \geq 1$. Then

$$\mu(u) \equiv 0 \bmod 2k(u)$$

Proof:

If $k(u) \geq 1$, there exist $2k(u)$ sectors. By lemma 6 each of these is symmetric to its neighboring sectors. Therefore $\mu(u)$ must be an integral multiple of $2k(u)$, as proposed.□

Conics

In this chapter we want to make some investigations on conics, especially on those which are factors of harmonic polynomials.

Centers, saddles and the location of conics

To describe the possible situation we define two distinguished points, namely the *center* and the *saddle* of a conic.

Every homogeneous polynomial P of degree two in $\mathbb{R}^3$ can be written as the quadratic form $P(x) = \langle x, A_P x \rangle$. The three eigenvectors e_i to the eigenvalues λ_i of the symmetric matrix A_P are orthogonal to each other and these are — normalized to one — the stationary points of the function p (at these points the gradiant of P is a multiple of the point itself and hence perpendicular to the sphere). Since we are interested in the factors of spherical harmonics, $p^{-1}(0) \neq \emptyset$ and P must change sign at its nodal set. Hence there must be a positive and a negative eigenvalue of A_P and we can assume without loss that $\lambda_1 < 0$, $\lambda_3 \geq \lambda_2 \geq 0$. If $\lambda_2 = 0$, than p contains a singular point and hence is reducible, i.e. it consists of a pair of lines (This follows immediately from Harnacks theorem, since then $\zeta(p) + \nu(p) = 3$).

If $\lambda_2 \neq \lambda_3$, then we call e_2 the *saddle point* of the conic p. (From Morse theory we know, that this really is a saddle point of the function p.)
If $\lambda_2 \neq 0$, i.e. p is not irreducible, than e_1 lies in the inside of the "oval" and we call it the *center* of the conic p.

Now we can use these points to describe the location of two factors of spherical harmonics, which have four common points.

Proposition 1:
Let p and q be two irreducible conics which have four different real common points. Suppose p and q are orthogonal to each other, i.e. their tangents at these points are orthogonal. Then (without loss) the center of p is the saddle of q and the saddle of p must be inside of q. Furthermore neither p nor q is a circle.
If moreover the stationary points of p and q coincide, then the center of one is the saddle of the other and vice versa.

Corollary 1:

If two conics have a common center, they cannot intersect orthogonally.

Proof:

It can easily be seen from symmetry, that the two conics have either four or no common points. By proposition 1 the two conics cannot intersect each other orthogonally in four points.□

Corollary 2:

If two conics have four common points, their union is connected.

Proof:

This follows immediately from proposition 1.□

Proposition 2:

If three conics p, q_1 and q_2 intersect each other orthogonally, where p intersects each of the other conics q_i in four points, then q_1 and q_2 have no common points.

The proofs of propositions 1 and 2 are given in appendix A, since they are rather long and technical.

Products of linear and quadratic factors

We want to use the above propositions, to give an upper bound for the number of nodal domains for the special class of polynomials, which consist of linear and quadratic factors only, and where all the quadratic factors intersect orthogonally.

First assume a polynomial P, that consists of k quadratic factors only, i.e.

$$P(x) = \prod_{j=1}^{k} P_j(x)$$

where the P_j are polynomials of degree two, which intersect each other orthogonally. Assume further that each of these conics p_j intersects any other in at most two points. Thus P has at most $k(k-1)$ (pairs of antipodal) singular points and thus

$$\nu(p) \leq 2k(k-1) \tag{1}$$

Now if each conic p_j intersects at most one other conic in four points, then we obviously have at most k more singular points of p.

If there is a conic p_1, which intersects each of two other conics p_2 and p_3 orthogonally in four points, p_2 and p_3 cannot have a common point by proposition 2. Hence the total number of singular points cannot exceed $k(k-1)+k$. Hence we have

Lemma 1:
Let P the product of k polynomials of degree two, which intersect orthogonally, then

$$\nu(p) \leq 2k^2 \tag{2}$$

Since $\zeta(p) > 2j$ if and only if there are j groups of conics, such that no conic of the one group intersects a conic of another group, we find by corollary 2 and Euler's theorem

Corollary 3:
Let U be a harmonic polynomial of even degree ℓ consisting of quadratic factors only, which intersect each other orthogonally, then

$$\mu(u) \leq \frac{\ell^2}{2} + 2$$

Now assume a conic p with equation

$$P(x) = \langle x, A_p x\rangle = \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 = 0 \tag{3}$$

with $\alpha \geq \beta \geq 0, \gamma < 0$. Its stationary points are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Now assume a line $L(x) = \langle a, x\rangle = a_1x_1 + a_2x_2 + a_3x_3 = 0$ intersects p orthogonally, that is $L'(x) = \langle \nabla L, \nabla P(x)\rangle = \langle a, A_p x\rangle = 0$ at the common points of l and p. Hence the lines l and l' must be equal and thus $L' = \nu L$ for some $\nu \in \mathbb{R}$. Hence by comparison of coefficients

$$\alpha a_1 = \nu a_1$$
$$\beta a_2 = \nu a_2$$
$$\gamma a_3 = \nu a_3$$

Obviously this equations hold if and only if $a = (1,0,0)$ or $a = (0,1,0)$, or p is circle and $a = (a_1, a_2, 0)$. Thus we get

Lemma 2:

If a straight line l intersects a conic p orthogonally, then either

(1) l is an axis of the conic p, i.e. a straight line through the vertices of the conic (= through its center and another eigenvalue of A_P); or

(2) p is a circle and l any straight line through its center.

From this we immediately have

Lemma 3:

If a conic p is not a circle, there are at most two straight lines, which intersect p orthogonally.

We want to use these two lemmata to study harmonic products of linear and quadratic factors. We show

Theorem 1:

Let u be a spherical harmonic that contains linear and quadratic factors only. If there is no singular point of u of multiplicity 3 or higher on any quadratic factor, then

$$\mu(u) \leq \begin{cases} \frac{1}{2}(\ell+1)^2 & \text{if } \ell \text{ is odd} \\ \\ \frac{1}{2}\ell(\ell+2) & \text{if } \ell \text{ is even} \end{cases} \tag{4}$$

Proof:

We use the propositions 1 and 2 and the above lemmata to calculate the possible number of intersection points between conics and straight lines. Because of corollary 3 and the theorem 1 about straight lines we can assume, that there is at least one line and one conic.

Since the existence of more than two components of u decreases the number of intersection points between conics, we can assume, that the nodal set of u is connected. To make the following more legible, we break the proof into several steps and distinguish between the type of the u_{lin} — the product of all linear factors of u (see the chapter about straight lines and theorem 1 there).

First we summarize some facts which can easily be deduced from the above results.

(I) A straight line, that intersects a conic p orthogonally, contains the center and a second stationary point of p (lemma 2).

(II) If two conics intersect the same two straight lines, then they must have a common center by lemma 2 and hence they have no common points by corollary 1.

(III) If a conic intersects three straight lines, it is a circle (lemma 3) and thus it cannot have four common points with any other conic by proposition 1.

(IV) The union of two conics, which have four common points, intersects each straight line by corollary 2.

(V) The nodal set of u is symmetric with respect to each of its straight lines (lemma 6 about straight lines).

Now we want to distinguish between the following cases (see theorems 1 and 2 about straight lines):

Case (A1): u has exactly one straight line.
Case (A2): u has exactly two straight lines.
Case (A3): u_{lin} is of type (a) and has more than two lines.
Case (B1): u_{lin} is of type (b) and has three lines.
Case (B2): u_{lin} is of type (b) and has more than three lines.
Case (C): u_{lin} is of type (c).
Case (D): u_{lin} is of type (d).
Case (E): u_{lin} is of type (e).

(These types are explained in theorem 1 about straight lines.)

For abbreviation we will use the following symbols:

- k total number of conics
- j number of conics, which intersect no or one straight line (I-conics[1])
- m number of conics, which intersect two ore more straight lines (M-conics[1])

Obviously $m = k - j$, $0 \leq k < \frac{\ell}{2}$ and $0 \leq j, m \leq k$. $\ell - 2k$ gives the number of straight lines.

The case (A1): ℓ is odd. By using lemma 1, Bezout's and Euler's theorem we have

$$\mu(u) \leq \frac{1}{2}(\ell-1)^2 + 2(\ell-1) + 2 = \frac{1}{2}(\ell+1)^2$$

[1] These names are used in this proof only. Thus forget them after reading this chapter and do not mix them up with "M-curves" used in literature.

The case (A3): There are at least three straight lines and all of them intersect in one point. Because of (III) and (IV) two conics cannot have four common points. Hence

$$\mu(u) \leq \mu(k,j) := \underbrace{2(\ell-2k-1)}_{\nu(u_{lin})} + \underbrace{4(\ell-2k)(k-j)}_{(\clubsuit)} + \underbrace{4j}_{(\diamondsuit)} + \underbrace{2j(j-1)}_{(\heartsuit)} + \underbrace{4j(k-j)}_{(\spadesuit)} + 2$$

where ($\clubsuit$) gives the number of intersection points of straight lines and M-conics, ($\diamondsuit$) of lines and I-conics; ($\heartsuit$) the number of common points of I-conics (due to (1)) and ($\spadesuit$) the number of intersection points of I-conics and M-conics. Due to (II) M-conics cannot have common points. (The number 2 is because of Euler's theorem.)

We have to maximize this function. Since

$$\frac{\partial \mu}{\partial j} = 4(-j+3k-\ell)+2$$

and since $j,k \in \mathbb{N}_0$ we have

$$\max_{0\leq j\leq k} \mu(k,j) = \begin{cases} \mu(k,0) = 2(\ell-2k-1)+4k(\ell-2k)+2 & \text{for } k < \frac{\ell}{3} \\ \mu(k,3k-\ell) = 2(5k^2+(1-4\ell)k+\ell^2-\ell) & \text{for } \frac{\ell}{3} \leq k < \frac{\ell}{2} \end{cases}$$

As can easily be seen (4) holds in the case $k < \frac{\ell}{3}$ (but this we already know from lemma 5 about straight lines). For the second case we remark, that $k \mapsto \mu(k, 3k-\ell)$ has not a local maximum and hence we just need to compute its value at $k = \frac{\ell}{3}$ and $k = \frac{\ell-3}{2}$ (for odd ℓ) or $k = \frac{\ell-4}{2}$ (for even ℓ) to show that (4) holds.

The case (B2): All but one straight lines intersect in the pole, one line is the equator. These lines cut out triangles from the sphere. Its vertices are the pole and two points on the equator.

First we remark, that there cannot be any two conics which have four common points by (III) and (IV).

Now a conic either intersects all lines through the pole (M_p-conic), and hence it is a circle by (III), or it intersects the equator and one meridian (M_e-conic) and thus its center is a vertex of a triangle. Because of symmetry (cf. (V)) there are at most two different types of such vertices. The first ones are the center of k_1 M_e-conics, the second ones the center of k_2 M$_e$-conics. Hence there are $\frac{\ell-2k-1}{2}k_1 + \frac{\ell-2k-1}{2}k_2 =: k_e$ M$_e$-conics and $(k-j-k_e)$ M$_p$-conics.

Now the M$_p$ cannot have common points. Furthermore M$_p$-conics and M$_e$-conics cannot have common points (otherwise such a pair of conics must have four). The M$_e$-conics, which are "neighboring" may intersect. Thus these may have $4k_1k_2(\ell - 2k-1)$ common points.

Therefore the following holds

$$\mu(u) \leq \underbrace{2(\ell-2k-2)+2(\ell-2k-1)}_{\nu(u_{lin})} + \underbrace{4(\ell-2k-1)(k-k_e-j)+8k_e+4j}_{(\spadesuit)}$$
$$+ \underbrace{2j(j-1)}_{(\diamondsuit)} + \underbrace{4k_1k_2(\ell-2k-1)}_{(\clubsuit)} + \underbrace{4j(k-j)}_{(\heartsuit)} + 2$$

where $(\spadesuit)$ gives the number of intersection points of the straight lines and the M$_p$-, M$_e$- and I-conics, respectively. $(\diamondsuit)$ gives the number of common points of I-conics (from (1)), $(\clubsuit)$ the number of common points of M$_e$-conics and $(\heartsuit)$ the number of intersection points of M- and I-conics.

Now if k_e is constant, then the right hand side becomes a maximum, if $k_1 = k_2 = \frac{k_e}{\ell-2k-1}$. Thus we have to maximize

$$\mu(k,j,k_e) = 4(\ell-2k-1) + 4(\ell-2k-1)(k-k_e-j) + 8k_e + 4j$$
$$+ 2j(j-1) + 4\frac{k_e^2}{\ell-2k-1} + 4j(k-j)$$

Since $k_e \mapsto \mu(k,j,k_e)$ has not a local maximum (since $\frac{\partial^2\mu}{\partial k_e^2} > 0$), we have

$$\max_{0\leq k_e\leq k-j} \mu(k,j,k_e) = \max\{\mu(k,j,0), \mu(k,j,k-j)\}$$

The case $k_e = 0$ is nearly the same as in case (A3) and hence it is not computed again. Thus we have to look at the function $\mu(k,j,k-j)$. By differentiating $\mu(k,j,k-j)$ with respect to j we see that the function $j \mapsto \mu(k,j,k-j)$ takes its maximum in

$$j = \frac{-4k^2+2k\ell-3\ell+3}{2(\ell-2k-3)}$$

for $k \leq \frac{\ell-4}{2}$. (For the moment we ignore that $j \in \mathbb{N}_0$.) Thus

$$\max_{0\leq j\leq k} \mu(k,j,k-j) \leq \underbrace{\frac{-8k^3+(12+4\ell)k^2+(34-28\ell)k+8\ell^2-23\ell+15}{2(\ell-2k-3)}}_{=:\mu'(k)}$$

When we differentiate the right hand side with respect to k we have

$$\frac{2(8k^3 + (12-8\ell)k^2 + (-18+2\ell^2)k - 18 + 18\ell - 3\ell^2}{(\ell - 2k - 3)^2}$$

Using Cardano's formula (see [VW1], p. 193, or [ME], p. 672) we can show, that the numerator has only one real root for $\ell \geq 3$. On the other hand this derivative of the numerator is less than zero at $k = 0$ for $\ell \geq 5$ and greater than zero at $k = \frac{\ell-4}{2}$ for all $\ell \geq 1$. The denominator always is positive. Hence the only local extremum of $\mu'(k)$ is a minimum. Therefore

$$\max_{0 \leq k \leq \frac{\ell-4}{2}} \mu'(k) = \max\{\mu'(0), \mu'(\tfrac{\ell-4}{2})\}$$

Computing $\mu'(0)$ and $\mu'(\frac{\ell-4}{2})$ shows that (4) holds for $\ell > 8$. At last we remind that $k_e > 0$ if it is a multiple of $(\ell - 2k - 1)$ which is not possible if $\ell < 10$. Computing the cases where $\ell = 6$, 7 or 8 finishes case (B2).

The case (B1): ℓ is odd, and there are (exactly) three lines, two meridians and the equator. Denote their intersection points with y_1, y_2 and y_3. Then any conic that intersects more than one line has a y_i as center. We denote their numbers with k_1, k_2 and k_3, respectively. By corollary 1 the conics with the same center cannot intersect. Because of symmetry two conics with centers y_i and y_j either have four or no common points. Hence because of proposition 1 (without loosing generality) only the conics with centers y_1 and y_2 may intersect. Because of (IV) any conic which intersects another conic in four points must have center y_i. Thus we have

$$\mu(u) \leq \underbrace{6}_{\nu(u_{lin})} + \underbrace{8(k_1+k_2+k_3)+4j}_{(\spadesuit)} + \underbrace{2j(j-1)}_{(\diamondsuit)} + \underbrace{8k_1k_2}_{(\clubsuit)} + \underbrace{4j(k_1+k_2+k_3)}_{(\heartsuit)} + 2$$

where ($\spadesuit$) gives the number of intersection points of the straight lines and the M-conics and I-conics, respectively. ($\diamondsuit$) gives the number of common points of I-conics, ($\clubsuit$) the number of common points of the M-conics and ($\heartsuit$) the number of intersection points of M- and I-conics.

Since $k_1 + k_2 + k_3 = k$, the right hand side is maximized (for constant ℓ, k,k_1,k_2 and j) by $k_3 = 0$. Thus using $k_2 = k - j - k_1$ we have to maximize

$$\mu(j, k_1) := 8 + 8(k-j) + 8k_1(k-j-k_1) + 2j(j-1) + 4j(k-j) + 4j$$

for $k = \frac{\ell-3}{2}$, $j + k_1 \leq k$. We arrive at

$$\max_{0 \leq k_1 \leq k-j} \mu(j, k_1) = \mu(j, \frac{k-j}{2}) = -6j + 2k^2 + 8k + 8$$

Since the right hand side is linearly in j

$$\max_{0 \leq j \leq k} \mu(j, \frac{k-j}{2}) = \mu(0, \frac{k}{2}) = 2k^2 + 8k + 8$$

Inserting $k = \frac{\ell-3}{2}$ we get $\mu(u) \leq \mu(j, k_1) \leq \frac{1}{2}(l+1)^2$ and (4) holds.

The case (A2): ℓ is even, and there are (exactly) two straight lines — meridians. Now the situation is similarly to the above case. But yet the equator is not a factor of u. Thus analogously to above we have

$$\mu(u) \leq 2 + \underbrace{8k_1 + 4(k-j-k_1)}_{(*)} + 8k_1(k-j-k_1) + 2j(j-1) + 4j(k-j) + 4j + 2$$

Mind that the k_2 M-conics having center y_2 do not intersect both straight lines $(*)$. Maximizing the right hand side and using $k = \frac{\ell-2}{2}$ we have

$$\mu(u) \leq \left[\frac{1}{2}(\ell+1)^2\right] = \frac{1}{2}\ell(\ell+2)$$

since ℓ is even, and (4) holds.

The case (C): There are six lines. Their location can be seen from figure 1 in the chapter about straight lines. These cut out identical triangles from the sphere (cf. (V)). The vertices of these triangles are the centers of the M-conis. Hence there are three types of M-conics. The number of which at each vertex is denoted by k_1, k_2 and k_3 respectively.

Analogously to case (B2) there are no two conics which have four common points. Moreover the M-conics of the same type cannot intersect. If two M-conics (of different type) intersect, their common points are in different triangles. By theorem 1 about straight lines there are 24 such triangles (12 on each hemisphere) and hence there are at most $(k_1k_2 + k_1k_3 + k_2k_3) \cdot 24$ common points. They intersect u_{lin} in at most $12(k_1 + k_2) + 8k_3$ points. Thus the following holds

$$\begin{aligned}\mu(u) \leq \underbrace{22}_{\nu(u_{lin})} &+ \underbrace{12(k_1+k_2) + 8k_3 + 4j}_{(\spadesuit)} + \underbrace{2j(j-1)}_{(\diamondsuit)} \\ &+ \underbrace{24(k_1k_2 + k_1k_3 + k_2k_3)}_{(\clubsuit)} + \underbrace{4j(k-j)}_{(\heartsuit)} + 2\end{aligned}$$

Now $k = j+4k_1+4k_2+6k_3$. Thus the right hand side becomes maximal for constant j, if $k_3 = 0$. Hence using $k_2 = \frac{k-j-4k_1}{4}$ we have to maximize the function

$$\mu(j,k_1) := 24 + 3(k-j) + 4j + 2j(j-1) + 6k_1(k-j-k_1) + 4j(k-j)$$

From this we easily have

$$\max_{k_1} \mu(j,k_1) = \mu(j, \frac{k-j}{2})$$

and

$$\max_{j} \mu(j, \frac{k-j}{2}) = \mu(k-1, \frac{1}{2}) = 2k^2 + 2k + \frac{49}{2}$$

Inserting $k = \frac{\ell-6}{2}$ shows, that (4) holds for $\ell \geq 7$.

The case (d): There are nine straight lines. Analogously to (c) we have to maximize the right hand side of

$$\mu(u) \leq \underbrace{46}_{\nu(u_{lin})} + \underbrace{16k_1 + 12k_2 + 8k_3 + 4j}_{(\spadesuit)} + \underbrace{2j(j-1)}_{(\diamondsuit)} + \underbrace{48(k_1k_2 + k_1k_3 + k_2k_3)}_{(\clubsuit)} + \underbrace{4j(k-j)}_{(\heartsuit)} + 2$$

to show that (4) holds. For doing this, use the fact, that $k = \frac{\ell-9}{2}$ and $k = j + 3k_1 + 4k_2$.

The case (e): There are fifteen straight lines. Analogously to (c) we have to maximize the right hand side of

$$\mu(u) \leq \underbrace{118}_{\nu(u_{lin})} + \underbrace{20k_1 + 12k_2 + 8k_3 + 4j}_{(\spadesuit)} + \underbrace{2j(j-1)}_{(\diamondsuit)} + \underbrace{120(k_1k_2 + k_1k_3 + k_2k_3)}_{(\clubsuit)} + \underbrace{4j(k-j)}_{(\heartsuit)} + 2$$

to show that (4) holds. Use that $k = \frac{\ell-15}{2}$ and $k = j + 6k_1 + 10k_2$.

Yet we have shown, that the theorem holds for all cases.□

Circles

We show a result analogously to lemma 2 about straight lines.

Lemma 4:
Let u be a spherical harmonic, that contains a latitude circle p (which is not the equator), and

$$u(\theta,\varphi) = \sum_{i=-\ell}^{\ell} \alpha_i Y_\ell^i(\theta,\varphi)$$

Then for each i either $\alpha_i = 0$ or p is a factor of $Y_\ell^i(\theta,\varphi)$

Proof:
Analogously to the proof of lemma 2 about straight lines, we use the linear independency of the trigonometric functions to show that each of the base functions $Y_\ell^i(\theta,\varphi)$ contains the circle or $\alpha_i = 0$.□

If it could be verified[2] that the zeros — except at -1, 0 and 1 — of the associated Legendre polynomials to a given ℓ are all different, then the only spherical harmonics containing latitude circles are base functions $Y_\ell^i(\theta,\varphi)$ (which may be rotated with respect to the pole). Numerically we can show that this is true for at least $\ell \leq 6$.

Conics and Straight Lines

Assume a point x on the conic p with equation

$$P(x) = \langle x, A_p x\rangle = \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 = 0$$

and an arbitrary point y on the sphere different from x. Let l_{xy} be the straight line through x and y. Then define $Q(x,y) = \frac{1}{2}\nabla L_{x,y} \cdot \nabla P(x)$. Since

$$L_{xy}(x') = \begin{vmatrix} x_1' & x_2' & x_3' \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

we have $\nabla L_{xy} = y \times x$ (where $\times$ denotes the cross product) and $\nabla P(x) = 2\, A_p x$ we have

$$Q(x,y) = \begin{vmatrix} y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \\ \alpha x_1 & \beta x_2 & \gamma x_3 \end{vmatrix}$$

which is linear in y and of degree two in x.
Now for all the points $\bar{x}$, where the straight line through $\bar{x}$ and y is orthogonal to the conic at $\bar{x}$, we must have $\nabla L_{\bar{x}y} \cdot \nabla P(\bar{x}) = 0$. Hence these points must be the common points of the curves p and q.

[2] I could not find such a result in literature.

From this we immediately get by Bezout's theorem

Lemma 5:
Let p be any proper conic with non-empty image and y be an arbitrary point on the sphere not on the conic. Then either p is a circle with center y, or there are at most four points $x_i \in p$, such that the straight line through y and x_i is perpendicular to the tangent on p at x_i, for each i.

If we choose the center of p as the point y, the situation of lemmata 1 and 2 occurs.

The angle ϵ between l_{xy} and the conic p can be calculated by means of the scalar product of ∇L_{xy} and $\nabla P(x)$. We get

$$\cos(\epsilon) = \frac{\nabla L_{xy} \cdot \nabla P(x)}{|\nabla L_{xy}|\,|\nabla P(x)|}$$

The numerator as well as the denominator of the rational function depend continuously on x. Since moreover $|\nabla L_{xy}|\,|\nabla P(x)| \neq 0$, $\cos(\epsilon)$ is a continuous function in x. Furthermore it cannot be constant unless p is a circle by lemma 2. Hence we have shown

Lemma 6:
Let y be the center of a conic p and x any point on p. Then the cosine of the angle at x between p and the straight line through x and y is a continuous function of x.

Quadratic factors

This subchapter we need for our study of the spherical harmonic of degree 6. The results, which are developed here, will be used to derive an upper bound for the number of nodal domains, where u has a factor of degree 2.

In general it is not easy to determine whether a conic is factor of any function of a given class of special polynomials. In our case we can use the harmonicity to give a suitable method for this.

Let U be a harmonic polynomial which contains an irreducible quadratic factor Q. Without loss we can assume

$$Q = \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2.$$

Then there exists a polynomial P of degree $\ell - 2$,

$$P = \sum_{i+j+k=\ell-2} a_{ijk} x_1^i x_2^j x_3^k$$

such that $U = Q \cdot P$. But since U is harmonic, we have

$$\Delta U = 0$$

Thus all coefficients of the polynomial ΔU must be zero. Now assume α, β and γ are known. Then the a_{ijk} can be calculated from these. As can easily be seen, they must satisfy the linear equation

$$A(\alpha, \beta, \gamma) \cdot \vec{a} = 0$$

where $A(\alpha, \beta, \gamma)$ is a matrix, depending on α, β and γ only, and $\vec{a}$ contains the coefficients a_{ijk}. Since at least one of these coefficient must be different from zero (otherwise $U \equiv 0$), $\det(A(\alpha, \beta, \gamma)) = 0$.

Hence the coefficients of Q must be points of the algebraic curves $\det(A(\alpha, \beta, \gamma)) = 0$, the coefficients of P must be in the kernel of $A(\alpha, \beta, \gamma)$. We have

$$(\alpha, \beta, \gamma) \in \left(\det(A(\alpha, \beta, \gamma))\right)^{-1}(0)$$

$$\vec{a} \in \ker(A(\alpha, \beta, \gamma))$$

We can simplify the search for quadratic factors of harmonic polynomials by using the following method:

Any harmonic polynomial U can be written as sum of four harmonic polynomials U_{ij} which are symmetric and antisymmetric with respect to the axes of the conic q (if p is a circle we can use any two meridians), i.e.

$$U = U_{00} + U_{01} + U_{10} + U_{11}$$

where

$$U_{ij}\big((-x, y, z)\big) = (-1)^i U_{ij}\big((x, y, z)\big)$$

$$U_{ij}\big((x, -y, z)\big) = (-1)^j U_{ij}\big((x, y, z)\big)$$

Now if U has the factor Q. Then it can easily be verified, that U_{ij} either contains the factor Q or is zero.

From now on let $\ell = 6$. Assume $U_{10} \neq 0$. Then

$$U_{10} = xz(\alpha x^2 + \beta y^2 + \gamma z^2)(\alpha' x^2 + \beta' y^2 + \gamma' z^2)$$

and

$$\begin{aligned}\Delta U_{10} = xz\big(&(20\alpha\alpha' + 2(\alpha\beta' + \alpha'\beta) + 6(\alpha\gamma' + \alpha'\gamma))x^2 \\ &+(12\beta\beta' + 6(\alpha\beta' + \alpha'\beta) + 6(\beta\gamma' + \beta'\gamma))y^2 \\ &+(20\gamma\gamma' + 6(\alpha\gamma' + \alpha'\gamma) + 2(\beta\gamma' + \beta\gamma))z^2\big) \\ &\equiv 0\end{aligned}$$

The algebraic curve $\det(A(\alpha,\beta,\gamma)) = 0$ is then given by

$$\Xi_{10} := \begin{vmatrix} 10\alpha + \beta + 3\gamma & \alpha & 3\alpha \\ \beta & \alpha + 2\beta + \gamma & \beta \\ 3\gamma & \gamma & 3\alpha + \beta + 10\gamma \end{vmatrix}$$

$$= \beta^3 + 13(\alpha+\gamma)\beta^2 + 35(\alpha^2+\gamma^2)\beta + 106\alpha\beta\gamma + 65\alpha\gamma(\alpha+\gamma) + 15(\alpha^3+\gamma^3)$$

Now if $U_{11} \neq 0$. Then

$$U_{11} = xy(\alpha x^2 + \beta y^2 + \gamma z^2)(\alpha' x^2 + \beta' y^2 + \gamma' z^2)$$

Thus we obtain Ξ_{11} by permuting y and z in U_{10} and hence by permuting β and γ in Ξ_{10}. Analogously for U_{01}. Therefore

$$\Xi_{11} := \gamma^3 + 13(\alpha+\beta)\gamma^2 + 35(\alpha^2+\beta^2)\gamma + 106\alpha\beta\gamma + 65\alpha\beta(\alpha+\beta) + 15(\alpha^3+\beta^3)$$

$$\Xi_{01} := \alpha^3 + 13(\beta+\gamma)\alpha^2 + 35(\beta^2+\gamma^2)\alpha + 106\alpha\beta\gamma + 65\beta\gamma(\beta+\gamma) + 15(\beta^3+\gamma^3)$$

It remains $u_{00} \neq 0$. Then

$$U_{00} = (\alpha x^2 + \beta y^2 + \gamma z^2)(ax^4 + by^4 + cz^4 + dx^2y^2 + ex^2z^2 + fy^2z^2)$$

And we get in the same way as above

$$\Xi_{00} := \begin{vmatrix} 15\alpha+\beta+\gamma & 0 & 0 & \alpha & \alpha & 0 \\ 0 & \alpha+15\beta+\gamma & 0 & \beta & 0 & \beta \\ 0 & 0 & \alpha+\beta+15\gamma & 0 & \gamma & \gamma \\ 6\beta & 6\alpha & 0 & 6\alpha+6\beta+\gamma & \beta & \alpha \\ 6\gamma & 0 & 6\alpha & \gamma & 6\alpha+\beta+6\gamma & \alpha \\ 0 & 6\gamma & 6\beta & \gamma & \beta & \alpha+6\beta+6\gamma \end{vmatrix}$$

$$
\begin{aligned}
= {} & 5\gamma^6 + \\
& (110\alpha + 110\beta)\gamma^5 + \\
& (635\alpha^2 + 2070\alpha\beta + 635\beta^2)\gamma^4 + \\
& (1060\alpha^3 + 8828\alpha^2\beta + 8828\alpha\beta^2 + 1060\beta^3)\gamma^3 + \\
& (635\alpha^4 + 8828\alpha^3\beta + 17010\alpha^2\beta^2 + 8828\alpha\beta^3 + 635\beta^4)\gamma^2 + \\
& (110\alpha^5 + 2070\alpha^4\beta + 8828\alpha^3\beta^2 + 8828\alpha^2\beta^3 + 2070\alpha\beta^4 + 110\beta^5)\gamma + \\
& (5\alpha^6 + 110\alpha^5\beta + 635\alpha^4\beta^2 + 1060\alpha^3\beta^3 + 635\alpha^2\beta^4 + 110\alpha\beta^5 + 5\beta^6)
\end{aligned}
$$

Summarizing this we have:

Lemma 5:
A conic $Q = \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2$ is factor of a harmonic polynomial of degree 6 if and only if $(\alpha : \beta : \gamma)$ is a point of the curve $\xi_{00} \cdot \xi_{10} \cdot \xi_{01} \cdot \xi_{11}$ of degree 15 (the curve determined by the equation $\Xi_{00} \cdot \Xi_{10} \cdot \Xi_{01} \cdot \Xi_{11} = 0$; figure 1).

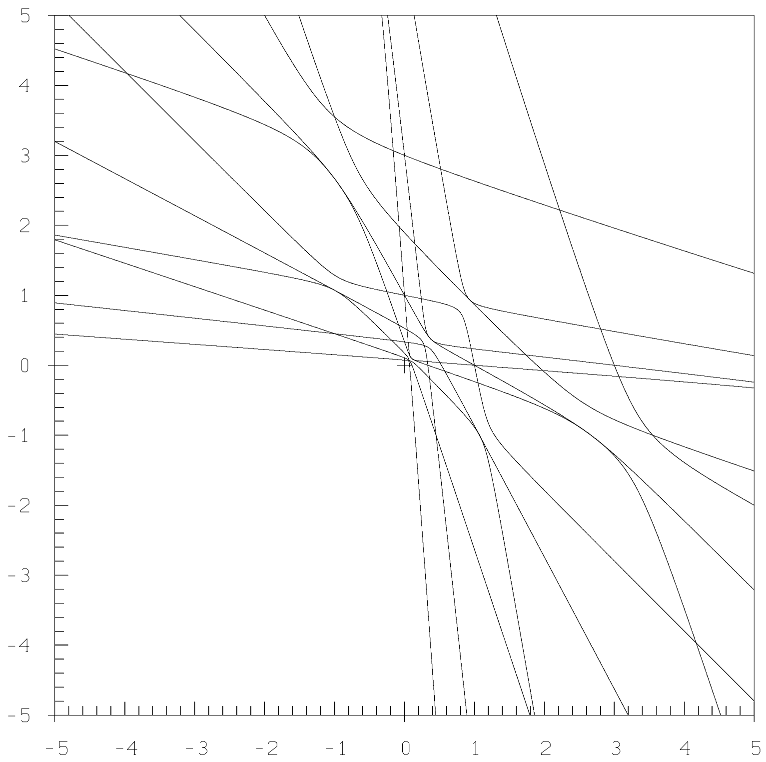


Figure 1: The curve $\xi_{00} \cdot \xi_{10} \cdot \xi_{01} \cdot \xi_{11}$ plotted for $\gamma = -1$, $-5 \leq \alpha, \beta \leq 5$.

A few sharp bounds

After we have done a lot of work, we are able to prove sharp bounds for a few special cases, namely the cases of spherical harmonics of degree 5 and 6. Our main tools are Harnack's theorem, the lemma about the union of curves and the results on straight lines and conics.

The result

Theorem 1:
Let u_ℓ be a spherical harmonic of degree ℓ. Then

(1) $\mu(u_1) = 2$	(4) $\mu(u_4) \le 12$
(2) $\mu(u_2) \le 4$	(5) $\mu(u_5) \le 18$
(3) $\mu(u_3) \le 8$	(6) $\mu(u_6) \le 24$

These bounds are sharp (see lemma 1 and the conjecture about spherical harmonics).

The proof for degree less than or equal to four

The case (1),(2) and (3) follow from Courant's nodal domain theorem.
Case (4) we already have shown in theorem 3 about algebraic curves.

Preliminaries for the proof for harmonic quintics and sextics

Before we start this we rewrite a few lemmata about algebraic curves.

Lemma 1:
Let p_m and p_n be factors of degree m and n of spherical harmonics. Then

$$\mu(p_m \cdot p_n) \le \mu(p_m) + \mu(p_n) + 2mn - \kappa'$$

where

$$\kappa' = \begin{cases} 2 & \text{if } p_m \text{ and } p_n \text{ have an invariant component} \\ 3 & \text{otherwise} \end{cases}$$

Equality holds if and only if

p_m and p_n have exactly mn common points,
$\varrho_{p_m}(x) = \varrho_{p_n}(x) = 1$ at each of these points and
$\zeta(p_m \cdot p_n) = \zeta(p_m) + \zeta(p_n) - 1$ if p_m or p_n has an invariant component
and $\zeta(p_m \cdot p_n) = \zeta(p_m) + \zeta(p_n) - 2$ otherwise.

Proof:

From the lemma about the union of two curves and since the equality holds in Euler's theorem we already have

$$\begin{aligned}\mu(p_m \cdot p_n) &= \nu(p_m \cdot p_n) + \zeta(p_m \cdot p_n) + 1 \\ &\leq (\nu(p_m) + \zeta(p_m) + 1) + (\nu(p_n) + \zeta(p_n) + 1) - 2 + 1 \\ &\qquad + 2mn - (\kappa' - 1) \\ &= \mu(p_m) + \mu(p_n) + 2mn - \kappa'\end{aligned}$$

where κ' is an appropriate integer which we have to estimate. From the lemma about the union of two curves we moreover know that $\kappa' - 1 \geq 1$, hence $\kappa' \geq 2$. Now suppose there is no invariant component and that a component C_m of p_m cuts the component C_n of p_n. (If there are no such components then $\kappa' > 2$ as can easily be checked.) Then the antipodal of C_m must cut the antipodal of C_n, too Hence we have for the number of components $\zeta(p_m \cdot p_n) \leq \zeta(p_m) + \zeta(p_n) - 2$. From this the proposition follows.

If p_m has not an invariant component, but p_n does, then the invariant component of p_n cuts not only a component C_m of p_m but also its antipodal. Again $\zeta(p_m \cdot p_n) \leq \zeta(p_m) + \zeta(p_n) - 2$.□

Lemma 2:

Let p be an irreducible factor of degree n of a spherical harmonic. Then

$$\mu(p) \leq (n-1)(n-2) + \kappa$$

where

$$\kappa = \begin{cases} 2 & \text{if } p \text{ has an invariant component} \\ 3 & \text{otherwise} \end{cases}$$

Proof:

It immediately follows from lemma 4 about algebraic curves, because in its proof we did not really use the degree of p but the possible existence of an invariant component.□

Lemma 3:

Let p_m and p_n be factors of degree m and n of spherical harmonics. Then

$$\mu(p_m \cdot p_n) \leq (m+n-1)(m+n-2) + 2 + \kappa''$$

where

$$\kappa'' = \begin{cases} 2 & \text{if } p_m \text{ or } p_n \text{ has an invariant component} \\ 3 & \text{otherwise} \end{cases}$$

Proof:

If we insert the result of lemma 2 into the right hand side of lemma 1 we arrive after a short calculation at $\mu(p_m \cdot u_n) \leq (m+n-1)(m+n-2)+2+(\kappa_m+\kappa_n+\kappa')$. If we set $\kappa'' = \kappa_m + \kappa_n + \kappa'$, we have the desired result.□

Lemma 4:

Let u_n^{irr} be irreducible factors of degree n of spherical harmonics. Then

(1) $\mu(u_1^{irr}) = 2$	(4) $\mu(u_4^{irr}) \leq 9$
(2) $\mu(u_2^{irr}) \leq 3$	(5) $\mu(u_5^{irr}) \leq 14$
(3) $\mu(u_3^{irr}) \leq 4$	(6) $\mu(u_6^{irr}) \leq 23$

Proof:

The results immediately follows from lemma 2 and lemma 1 about algebraic curves.□

Now we are ready to prove the cases (5) and (6).

Degree five

We decompose u into its irreducible factors and distinguish between

Case (i): $u = u_1^{irr} \cdot u_1^{irr} \cdot u_1^{irr} \cdot u_1^{irr} \cdot u_1^{irr}$
Case (ii): $u = u_2^{irr} \cdot u_1^{irr} \cdot u_1^{irr} \cdot u_1^{irr}$
Case (iii): $u = u_2^{irr} \cdot u_2^{irr} \cdot u_1^{irr}$
Case (iv): $u = u_3^{irr} \cdot u_1^{irr} \cdot u_1^{irr}$
Case (v): $u = u_3^{irr} \cdot u_2^{irr}$
Case (vi): $u = u_4^{irr} \cdot u_1^{irr}$
Case (vii): $u = u_5^{irr}$

Now we use lemmata 1, 3 and 4 and lemma 1 about algebraic curves to show
Case (vii): $\mu(u_5^{irr}) \leq 14$
Case (vi): $\mu(u_4^{irr} \cdot u_1^{irr}) \leq (4+1-1)(4+1-2)+2+2 = 16$
Case (v): $\mu(u_3^{irr} \cdot u_2^{irr}) \leq (3+2-1)(3+2-2)+2+2 = 16$
Case (iv): $\mu((u_3^{irr} \cdot u_1^{irr}) \cdot u_1^{irr}) \leq ((3+1-1)(3+1-2)+2+2)+2+2 \cdot 4 \cdot 1-2 = 18$
Case (iii): $\mu(u_2^{irr} \cdot (u_2^{irr} \cdot u_1^{irr})) \leq 3+((2+1-1)(2+1-2)+2+2)+2 \cdot 2 \cdot 3-3 = 18$

For the **cases (i)** and **(ii)** we use theorem 1 about conics or alternatively lemma 5 about straight lines.

Altogether we have

$$\mu(u) \leq 18$$

as proposed.□

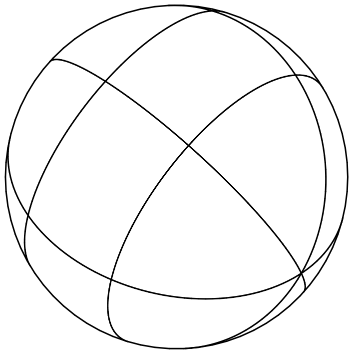


Figure 1: Spherical harmonic of degree 5 with 18 nodal domains

Degree six

Again we decompose u into its irreducible factors and distinguish between

Case (i): $u = u_1^{irr} \cdot u_1^{irr} \cdot u_1^{irr} \cdot u_1^{irr} \cdot u_1^{irr} \cdot u_1^{irr}$
Case (ii): $u = u_2^{irr} \cdot u_1^{irr} \cdot u_1^{irr} \cdot u_1^{irr} \cdot u_1^{irr}$
Case (iii): $u = u_2^{irr} \cdot u_2^{irr} \cdot u_1^{irr} \cdot u_1^{irr}$
Case (iv): $u = u_2^{irr} \cdot u_2^{irr} \cdot u_2^{irr}$
Case (v): $u = u_3^{irr} \cdot u_1^{irr} \cdot u_1^{irr} \cdot u_1^{irr}$
Case (vi): $u = u_3^{irr} \cdot u_2^{irr} \cdot u_1^{irr}$
Case (vii): $u = u_3^{irr} \cdot u_3^{irr}$
Case (viii): $u = u_4^{irr} \cdot u_1^{irr} \cdot u_1^{irr}$
Case (ix): $u = u_4^{irr} \cdot u_2^{irr}$
Case (x): $u = u_5^{irr} \cdot u_1^{irr}$
Case (xi): $u = u_6^{irr}$

Again we get from lemmata 1, 3 and 4 and lemma 1 about algebraic curves
Case (xi): $\mu(u_6^{irr}) \leq 23$
Case (x): $\mu(u_5^{irr} \cdot u_1^{irr}) \leq (5+1-1)(5+1-2)+2+2 = 24$

Case (vii) $\mu(u_3^{irr} \cdot u_3^{irr}) \leq (5+1-1)(5+1-2)+2+2 = 24$

All the other cases are more tricky.

Case (viii): From the above lemmata we have $\mu((u_4^{irr} \cdot u_1^{irr}) \cdot u_1^{irr}) \leq ((4+1-1)(4+1-2)+2+2)+2+2\cdot 5\cdot 1-2 = 26$. But we know from lemma 8 about straight lines that $\mu(u_6)$ must be a multiple of four. Hence $\mu(u_6) \leq 24$, as desired.

Case (vi): Again $\mu((u_3^{irr} \cdot u_2^{irr}) \cdot u_1^{irr}) \leq ((3+2-1)(3+2-2)+2+2)+2+2\cdot 5\cdot 1-2 = 26$. Now by lemma 7 about straight lines u_6 must be antisymmetric with respect to the center of the hemisphere cut out by the line u_1^{irr}. Hence the number of nodal domains of u_6 in this hemisphere must be even. Both hemispheres must be equal because of inversion symmetry, the total number of nodal domains of u_6 must be a multiple of four. Again $\mu(u_6) \leq 24$.

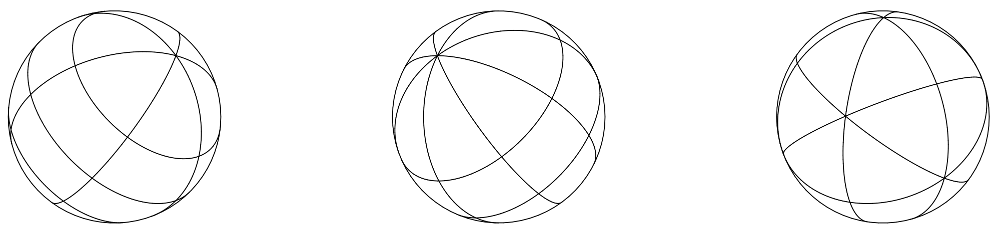


FIGURE 2: SPHERICAL HARMONICS OF DEGREE 6 WITH 24 NODAL DOMAINS

For the **cases (i)** and **(ii)** we have by theorem 1 about conics $\mu(u_6) \leq 24$.

Case (v): We either have $k(u_6) = 3$, where $k(u)$ denotes the number of meridians, and hence $\mu(u_6) \leq 24$ by lemma 5 about straight lines, or $k(u_6) = 2$. But by lemma 2, theorem 1, lemma 4, lemma 2 about straight lines and lemma 1 about spherical harmonics this case does not occur.

Case (iii): Since there are just two conics, they either intersect orthogonally or they have no common point. Hence by theorem 1 about conics the result follows.

Case (iv): We have $\mu((u_2^{irr} \cdot u_2^{irr}) \cdot u_2^{irr}) \leq ((2+2-1)(2+2-2)+2+3)+3+2\cdot 4\cdot 2-3 = 27$.

If at most two conics intersect in a point, $\mu(u_6) \leq 24$ follows from theorem 1 about conics.

If (and only if) there is exactly one point, which is common to all three conics, and if there is no invariant component of u_6, then we get by this method $\mu(u_6) \leq 25$. But[1] then one component of one conic intersects just one component of the others. We then would get an convex octagon with angle sum less than $3\frac{\pi}{2} + \frac{2\pi}{3} + 4\pi < 6\pi$, a contradiction since the angle sum of an octagon on a sphere must be greater than 6π (see figure 3).

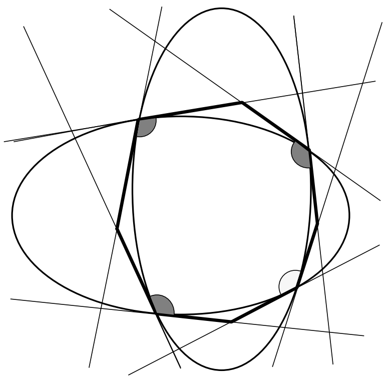


FIGURE 3: THE THREE MARKED ANGLES OF THE OCTAGON MUST BE $\frac{\pi}{2}$, THE FOURTH ONE $\frac{2\pi}{3}$.

Case (ix): From lemma 3 we have $\mu(u_4^{irr} \cdot u_2^{irr}) \leq 25$. To improve this estimate, we need a whole subchapter (see below).

Hence we have shown (or will show) that

$$\mu(u_6) \leq 24$$

in all possible cases, and we are done.□

The final case (ix)

The symmetric case

First we deal with the case, where u_4^{irr} is symmetric with respect to the axes of u_2^{irr}. Under this circumstances we can use lemma 5 about conics to show that our

[1] Alternatively we can use a method similar to that of case (ix).

proposition holds. It turns out that nearly all harmonic products of conics and quartics are symmetric. The few exceptions can be calculated explicitly.

We already know from lemma 3, $\mu(u_4^{irr} \cdot u_2^{irr}) \leq 25$. By lemma 1 this only is possible

if $u_4^{irr} \cdot u_2^{irr}$ has not an invariant component,

if just one pair of antipodal components of u_4^{irr} intersects with u_2^{irr} — we denote (the union of these) components with CC (critical component),

if u_4^{irr} and u_2^{irr} have 8 different common points, which are not singular points of u_4^{irr},

if u_4^{irr} has 9 nodal domains,

and hence if u_4^{irr} has no singular points of multiplicity 3 or higher (Harnack's theorem and the theorem about the genus of a curve).

We will show that this is impossible.

(I) First we note, that there must be a component C of u_4^{irr}, which does not intersect the conic u_2^{irr}. Otherwise u_4^{irr} has 3 singular points which either are located on one axis of the conic and hence u_4^{irr} would be decomposable by Bezout's theorem. Or these are the stationary points of the conic, hence orthogonal and thus u_4^{irr} consists of one invariant component only. Both contradict to our assumptions.

(II) Now take an arbitrary point a from the inside of this component. Each line through this point cuts this components at least twice and hence by Bezout's theorem it intersects the critical component CC in at most two points.

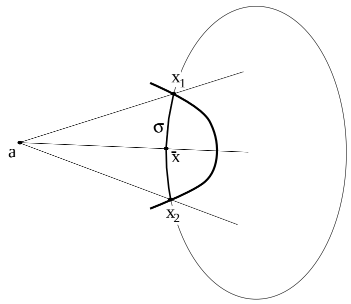


FIGURE 4

(III) As can easily be seen, C cannot be inside of CC. Otherwise there neither are other components besides C and CC (and their antipodal sets), nor there are any singular points of u_4^{irr}. Thus u_4^{irr} would not have nine nodal domains.

(IV) Because of (II) and (III) we can split CC into two connected arcs, such that both intersect any straight line through the point a (inside of C) at most once. Now take two common points x_1 and x_2 of one arc and the conic, which are "neighboring" on the conic (see figure 4).

(V) The straight lines through a and the common points x_1 and x_2 cannot be orthogonal to the conic at these points. Otherwise they must be inflectional tangents to u_4^{irr}, a contradiction to (II).

(VI) Let σ denote the arc on the conic between x_1 and x_2, and l_y the line through a and an arbitrary point on σ. Then the angle between the tangent of the conic at y and l_y depends continuously on y (lemma 6 about conics). Assume each line l_y intersects σ only once. Then the cosines of this angle at x_1 and x_2 have different sign and hence there exists a point $\bar{x} \in \sigma$, where $l_{\bar{x}}$ intersects the conic orthogonally (see figure 4).

(VII) Because of the symmetric properties all common points, which are neighbored on the conic, are neighbored on u_4^{irr} (i.e. there exists an arc on the curves which only contains these common points) (see figure 5).

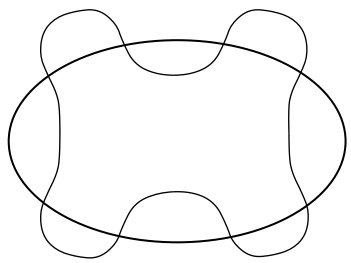


FIGURE 5: A SITUATION THAT FULFILLS OUR ASSUMPTIONS

(VIII) As can easily be seen, there are at most two common points, where the arc σ (defined in (VI)) intersects an straight line l_y more than once.

(IX) There are eight common points of u_4^{irr} and the conic. Because of (VI), (VII) and (VIII) there exist at least 6 points $\bar{x}_i$ on the conic, where the straight lines $l_{\bar{x}_i}$ intersect orthogonally. But this is a contradiction to lemma 5 about conics.

The non-symmetric case

In the last subchapter of the chapter about conics we have deduced the properties of a spherical harmonic u that contains a conic $q = u_2^{irr}$ with equation $Q = \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2$. If

$$U = U_{00} + U_{01} + U_{10} + U_{11}$$

where

$$U_{ij}\big((-x, y, z)\big) = (-1)^i U_{ij}\big((x, y, z)\big)$$

$$U_{ij}\big((x, -y, z)\big) = (-1)^j U_{ij}\big((x, y, z)\big)$$

then either U_{ij} also contains this conic, or $U_{ij} = 0$. Obviously u is not (anti-)symmetric with respect to both axes of the conic, if and only if at least two of U_{ij} are not equal to zero. Hence we have to compute the common points of the curves ξ_{ij} (see the chapter about conics).
Because of lemma 4 about conics we only are interested in such conics, which are not circles. Furthermore we are interested in irreducible conics only, that is, all coefficients q must not be zero. Therefore we can assume

$$\alpha, \beta > 0,\ \alpha \neq \beta \text{ and } \gamma < 0 \tag{1}$$

Figure 6 shows the curves ξ_{00}, ξ_{10} and ξ_{11}, which are given by

$$\begin{aligned}
\Xi_{00} = & 5\gamma^6 + \\
& (110\alpha + 110\beta)\gamma^5 + \\
& (635\alpha^2 + 2070\alpha\beta + 635\beta^2)\gamma^4 + \\
& (1060\alpha^3 + 8828\alpha^2\beta + 8828\alpha\beta^2 + 1060\beta^3)\gamma^3 + \\
& (635\alpha^4 + 8828\alpha^3\beta + 17010\alpha^2\beta^2 + 8828\alpha\beta^3 + 635\beta^4)\gamma^2 + \\
& (110\alpha^5 + 2070\alpha^4\beta + 8828\alpha^3\beta^2 + 8828\alpha^2\beta^3 + 2070\alpha\beta^4 + 110\beta^5)\gamma + \\
& (5\alpha^6 + 110\alpha^5\beta + 635\alpha^4\beta^2 + 1060\alpha^3\beta^3 + 635\alpha^2\beta^4 + 110\alpha\beta^5 + 5\beta^6)
\end{aligned}$$

$$\Xi_{10} = \beta^3 + 13(\alpha + \gamma)\beta^2 + 35(\alpha^2 + \gamma^2)\beta + 106\alpha\beta\gamma + 65\alpha\gamma(\alpha + \gamma) + 15(\alpha^3 + \gamma^3)$$

$$\Xi_{11} := \gamma^3 + 13(\alpha + \beta)\gamma^2 + 35(\alpha^2 + \beta^2)\gamma + 106\alpha\beta\gamma + 65\alpha\beta(\alpha + \beta) + 15(\alpha^3 + \beta^3)$$

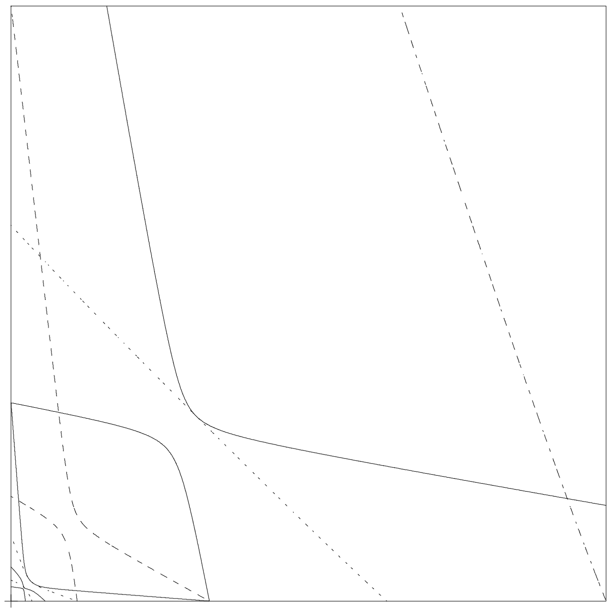


FIGURE 6: THE CURVES ξ_{00} (FULL LINES), ξ_{10} (DASHED LINES) AND ξ_{11} (DOTTED LINES), PLOTTED FOR $\gamma = -1$, $0 \le \alpha, \beta \le 3$. WE ARE INTERESTED IN THE INTERSECTION POINTS WHICH LIE NEITHER ON THE BOUNDARY NOR ON THE DIAGONAL.

At first we show that ξ_{10} and ξ_{01} have not a common point that fulfills (1). The resultant of these polynomials with respect to γ is given by

$$R(\alpha, \beta) = const \cdot \alpha\beta(\alpha - \beta)^5(16\alpha^2 + 37\alpha\beta + 16\beta^2)$$

Obviously there is no root fulfilling (1).

Analogously the resultant of ξ_{00} and ξ_{11}, which is given by

$$\begin{aligned} R(\alpha, \beta) = const \cdot (\beta - \alpha)^8(\alpha + \beta)^2 & \\ & (25600\alpha^8 + 951680\alpha^7\beta + 8472760\alpha^6\beta^2 \\ & + 30649480\alpha^5\beta^3 + 48618941\alpha^4\beta^4 + 30649480\alpha^3\beta^5 \\ & + 8472760\alpha^2\beta^6 + 951680\alpha\beta^7 + 25600\beta^8) \end{aligned}$$

has not a root fulfilling (1), since all the coefficients of the factor of 8th degree are positive. Thus u_{00} and u_{11} cannot have a common conic.

Next we calculate the common points of ξ_{10} and ξ_{11}. Their resultant is given by

$$R(\alpha,\beta) = const\cdot\beta\,(3\alpha+8\beta)^3$$
$$(-1200\alpha^5 - 5275\alpha^4\beta - 3830\alpha^3\beta^2 - 795\alpha^2\beta^3 + 4\alpha\beta^4 + 8\beta^5)$$

The only roots, which maybe satisfy (1) are the roots of the factor of degree 5. Since $\beta > 0$ by condition (1), we can assume $\beta = 1$. Then $\alpha = \frac{-5+\sqrt{345}}{160}$ is the only positive root. Hence we have found one conic (with coefficients $\alpha = \frac{-5+\sqrt{345}}{160}$, $\beta = 1$. γ remains to be compute.), which (possibly) is the common factor of some u_{10} and u_{11}. That is, there exists a pencil of harmonic polynomials where all polynomials contain the same conic. But since u_{10} and u_{11} contain the line $x = 0$, all these harmonic polynomials are of case (vi).
What we have to do, is to show, that this pencil is not part of a net of polynomials containing the same conic.

Now we compute the common points of ξ_{00} and ξ_{10}. Their resultant is given by

$$R(\alpha,\beta) = const\cdot\beta^2\,(10\alpha+\beta)^4(16\alpha^2+16\alpha\beta+\beta^2)^2$$
$$(-105\alpha^8 - 1260\alpha^7\beta + 15370\alpha^6\beta^2 + 18972\alpha^5\beta^3 - 186993\alpha^4\beta^4$$
$$- 392840\alpha^3\beta^5 - 124248\alpha^2\beta^6 + 79488\alpha\beta^7 - 5120\beta^8)$$

The only roots, which maybe satisfy (1) are the roots of the irreducible factor P of degree 8.

By inserting the root of the resultant of ξ_{10} and ξ_{11} into P we immediately see, that there cannot be a net of harmonic polynomials having the same quadratic factor.

Because of the symmetric properties of ξ_{10}, ξ_{01} and ξ_{00} we just have to calculate the pencils of harmonic polynomials determined by the common points of ξ_{10} and ξ_{00} to get all non-symetric harmonic products of $u_2^{irr}\cdot u_4^{irr}$.

Since $\beta > 0$ by condition (1), we can assume $\beta = 1$. We have to find the positive roots of

$$-5120 + 79488\alpha - 124248\alpha^2 - 392840\alpha^3 - 186993\alpha^4$$
$$+18972\alpha^5 + 15370\alpha^6 - 1260\alpha^7 - 105\alpha^8 = 0$$

Unluckily the roots of this polynomial cannot be computed algebraically. By Newton's method we get the following approximation solutions for α:

$$\alpha_{-4} = -18.9789974945 \qquad \alpha_1 = 0.0755371323178$$
$$\alpha_{-3} = -1.97714788200 \qquad \alpha_2 = 0.252045262853$$
$$\alpha_{-2} = -1.93448215875 \qquad \alpha_3 = 5.42427064004$$
$$\alpha_{-1} = -1.05088743317 \qquad \alpha_4 = 6.18966193324$$

Only the positive roots fulfill condition (1).

Now we compute γ. We find only one negative value for γ for each α_i, which are

$$\gamma_1 = -1.977147882$$
$$\gamma_2 = -1.050887433$$
$$\gamma_3 = -1.934482159$$
$$\gamma_4 = -18.97899749$$

Now we compute the coefficient of the second factor of the polynomials U_{00} and U_{10}. For this we have to calculate the kernels of the matrices $A_{00}(\alpha_i, \beta = 1, \gamma_i)$ and $A_{10}(\alpha_i, \beta = 1, \gamma_i)$ (see the subchapter about quadratic factors in the chapter about conics). These have dimension one, which can be checked be calculating proper subdeterminants.

Let $P = \alpha' x_1^2 + \beta' x_2^2 + \gamma' x_3^2$, such that

$$U_{10} = xz(\alpha x^2 + \beta y^2 + \gamma z^2)(\alpha' x^2 + \beta' y^2 + \gamma' z^2)$$

Then

$$\alpha'_1 = 0.01201884918 \qquad \beta'_1 = 0.99388037056 \qquad \gamma'_1 = -0.10980599374$$
$$\alpha'_2 = 0.83511989341 \qquad \beta'_2 = -0.49423712193 \qquad \gamma'_2 = -0.24146310468$$
$$\alpha'_3 = 0.29874674313 \qquad \beta'_2 = 0.11808324865 \qquad \gamma'_3 = -0.94699880140$$
$$\alpha'_4 = 0.94482260353 \qquad \beta'_4 = 0.05768276758 \qquad \gamma'_4 = -0.32246386801$$

For the second factor of U_{00}, $P = ax^4 + by^4 + cz^4 + dx^2y^2 + ex^2z^2 + fy^2z^2$, we get

$$a_1 = -0.56062745441 \qquad b_1 = -0.04344483995 \qquad c_1 = -0.03058148292$$
$$d_1 = 0.64205475765 \qquad e_1 = 0.51508373463 \qquad f_1 = -0.07299733311$$

$$a_2 = -0.00115058850 \qquad b_2 = -0.06791524977 \qquad c_2 = -0.07526374479$$
$$d_2 = -0.02889197594 \qquad e_2 = 0.04591850195 \qquad f_2 = 0.99336715701$$

$$a_3 = 0.01603900715 \qquad b_3 = -0.04756488747 \qquad c_3 = -0.07241673797$$
$$d_3 = -0.10205886393 \qquad e_3 = -0.13576307628 \qquad f_3 = 0.98152357241$$

$$a_4 = 0.05257378345 \qquad b_4 = 0.52762921503 \qquad c_4 = 0.02609334688$$
$$d_4 = -0.71039616027 \qquad e_4 = 0.07449951308 \qquad f_4 = -0.45601498222$$

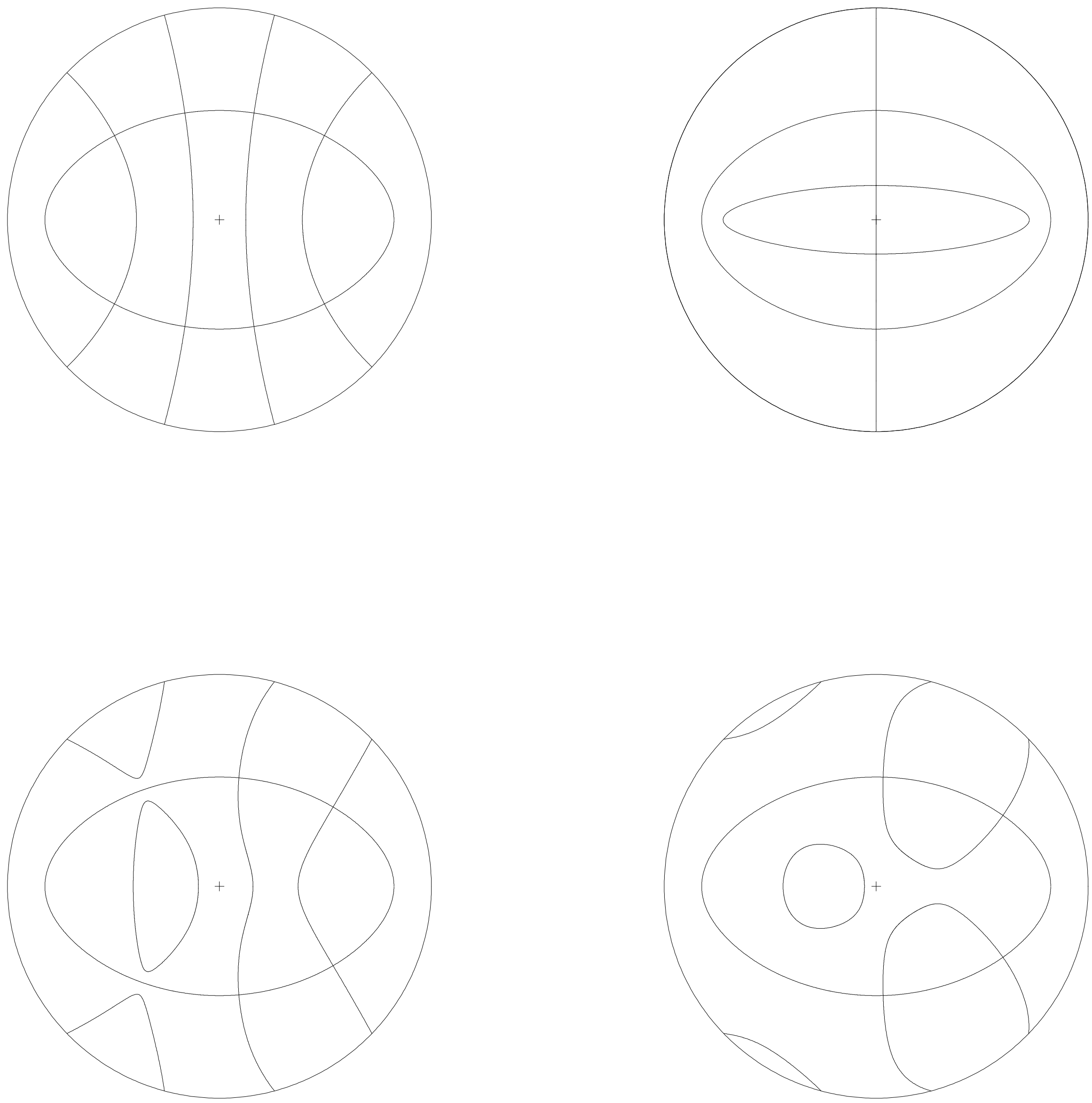


FIGURE 7: FOUR POLYNOMIALS OF THE PENCIL $u_1(\lambda)$ (STEREOGRAPHIC PROJECTION OF THE NORTHERN HEMISPHERE, THE CROSS MARKS THE POLE).

The result of these calculations is, that there are four pencils of harmonics polynomials, which contain a conic, but which are not symmetric with respect to the axes of this conic. These pencils are

$$U_i(\lambda) = (\alpha_i x^2 + \beta_i y^2 + \gamma_i z^2)\Big(\lambda \underbrace{xz(\alpha_i' x^2 + \beta_i' y^2 + \gamma_i' z^2)}_{=:U_i^1} + (1-\lambda)\underbrace{(a_i x^4 + b_i y^4 + c_i z^4 + d_i x^2 y^2 + e_i x^2 z^2 + f_i y^2 z^2)}_{=:U_i^2}\Big)$$

for $i = 1,2,3,4$ and $\lambda \in [0,1]$.
Figure 7 shows $u_1(0)$, $u_1(1)$ and two further spherical harmonics of the first pencil in stereographic projection.

Now look at figure 8. It shows u_1^1 (thick line), u_1^2 (dashed line) and the common conic of the pencil (thin line) in stereographic projection. The area, where u_1^1 and u_1^2 have different sign is hatched. Since the nodal set of $u_1(\lambda)$ depends continuously on λ the nodal set of $u_1'(\lambda) := \lambda u_1^1 + (1-\lambda)u_1^2$ must be in the interior[2] of this area for $\lambda \in (0,1)$ (except for the intersection points of u_1^1 and u_1^2, which always are points of these nodal sets).
Now start at $\lambda = 0$ — then $\mu(u_1'(0)) = 18$ — and look what happens if λ increases. Then no singular points occurs in the areas which are hatched vertically. Otherwise there must be a nodal domain of $u_1(\lambda)$ inside this area, a contradiction to the variation principle.
If a singular point occur in an area which is hatched horizontally, then firstly it is the only one and secondly it must lie on the common conic. Otherwise there would be a nodal domain of $u_1(\lambda)$ which is contained in a nodal domain of $u_1(0) = u_{00}$. Thus this additional singular points do not increase the number of nodal domains of $u_1(\lambda)$.
In the area which is hatched horizontally and vertically (at most) one singular point can occur. Now the number of nodal domains may increase by two. Together we have that the number of nodal domains in this pencil cannot exceed 18+2, what is less than 24.
The above proof does not need the exact location of the nodal lines of u_6 but only the relative location of the nodal lines of the factors u_1^1, u_1^2 and of the common

[2] cf. the small perturbation theorem in [VI] or [GU1].

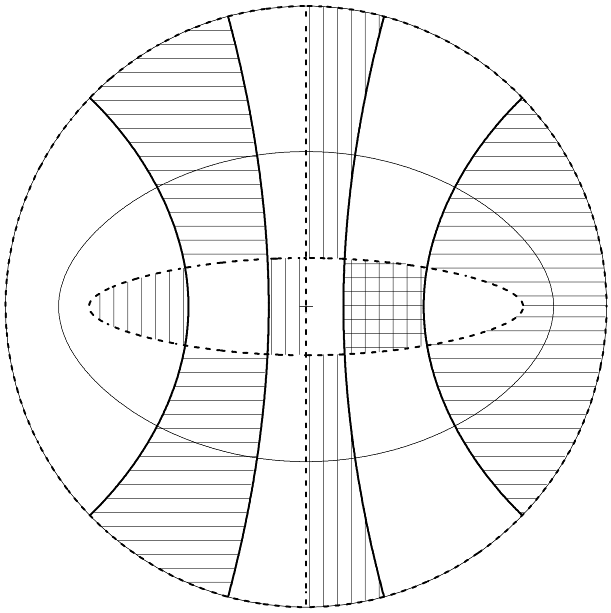


FIGURE 8: u_1^1 (THICK LINE), u_1^2 (DASHED LINE) (INCLUDES THE EQUATOR) AND THE COMMON CONIC OF THE PENCIL (THIN LINE) IN STEREOGRAPHIC PROJECTION.

conic. Since these curves are non-singular their numerically computed locations hardly differ from their exact ones. Thus $\mu(u_6) < 24$ in this case.

Figure 9 shows the situations for the three other pencils. For all this pencils we get $\mu(u_i(\lambda)) \leq 24$, which finishes the proof of theorem 1.$\square$

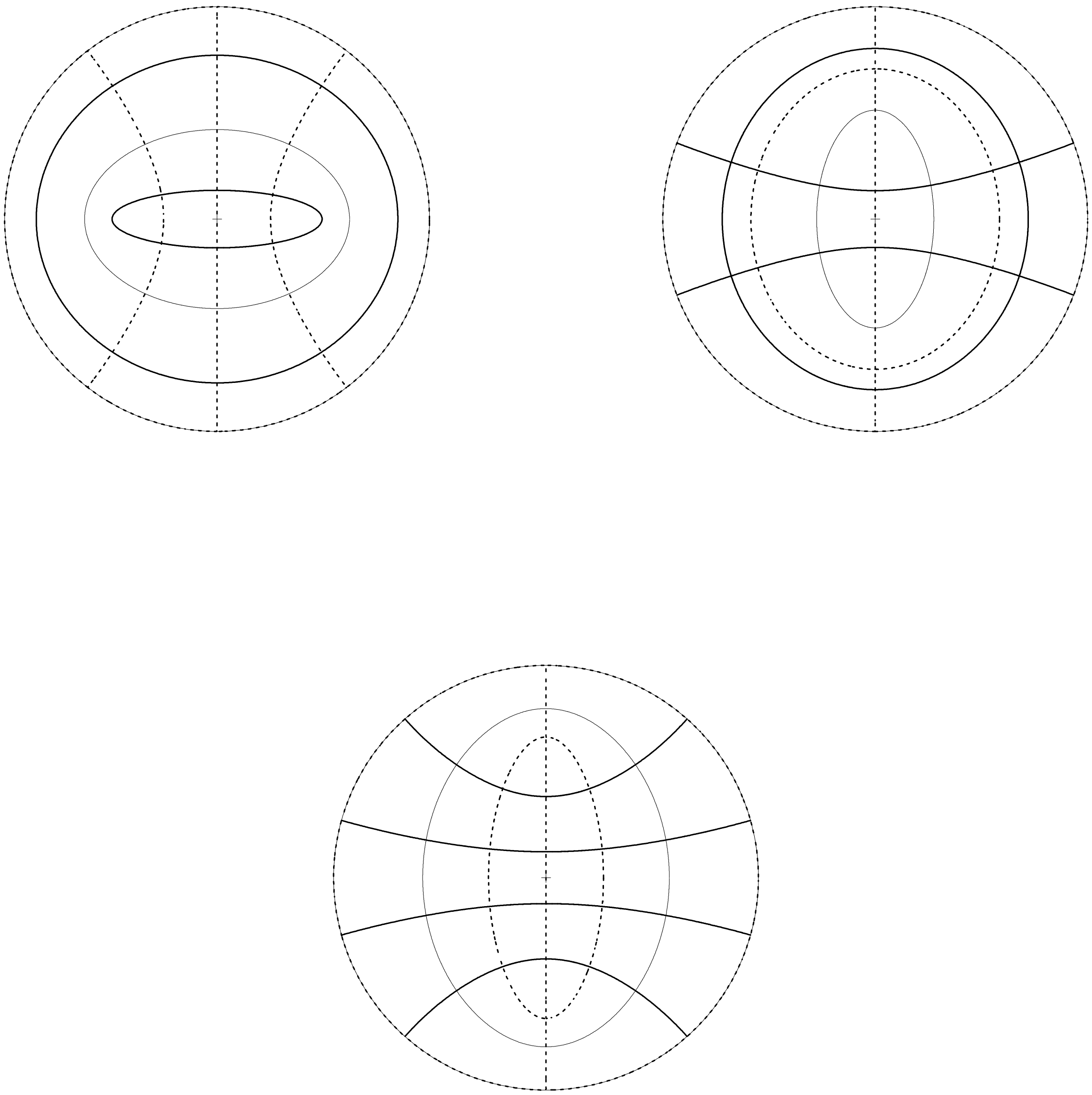


FIGURE 9: u_i^1 (THICK LINE), u_i^2 (DASHED LINE) (INCLUDES THE EQUATOR) AND THE COMMON CONIC OF THE i-TH PENCIL (THIN LINE) IN STEREOGRAPHIC PROJECTION FOR $i = 2,\ 3,\ 4$.

Non-singular curves

In this chapter we want to find estimatess on the number of nodal domains of non-singular spherical harmonics. First we show by construction that this number increases with the same order of ℓ as it does for arbitrary spherical harmonics. Then we deduce an estimate for this number which curiously is exactly the same we have derived from Harnack's theorem. But we are able to improve this a little bit. As a corollary we have, that no spherical harmonic can be a M-curve (except for $\ell \leq 3$).

A construction

Theorem 1:

(i) If $\ell = 2k$, $k \in \mathbb{N}$, then there exists a non-singular spherical harmonic of degree ℓ with $k^2 + k + 1$ nodal domains.

(ii) If $\ell = 2k - 1$, $k \in \mathbb{N}$, then there exists a non-singular spherical harmonic of degree ℓ with $(k-1)^2 + 2 + [(k-1) \bmod 2]$ nodal domains.

Proof:

Case (i): $\ell = 2k$

We denote the following spherical harmonics with

$$u_1 = P_\ell^k(\cos\theta) \cdot \sin(k\phi)$$
$$u_2 = P_\ell^\ell(\cos\theta) \cdot \cos(\ell\phi)$$
$$u_3 = P_\ell^0(\cos\theta)$$

The nodal set of u_1 consists of k meridians and k latitude circles. Hence $\mu(u_1) = \frac{\ell^2}{2} + \ell = 2k^2 + 2k$ and $\mu^-(u_1) = k^2 + k$, i.e. u_1 has $(k^2 + k)$ negative nodal domains.

The nodal set of u_2 contains ℓ meridians. u_1 and u_2 do not have a common straight line and u_2 is positive on all meridians (except at the pole) of u_1, since $\ell = 2k$ and $\cos x = \sin(x - \frac{\pi}{2})$ (see figure 1).

The singular points of u_1 are the pole and the intersection points of the meridians with the latitude circles. Since u_3 assumes a positive value at the pole, $u_2 + \varepsilon_1 u_3$ assumes positive values at all singular points of u_1 for sufficiently small $\varepsilon_1 > 0$.

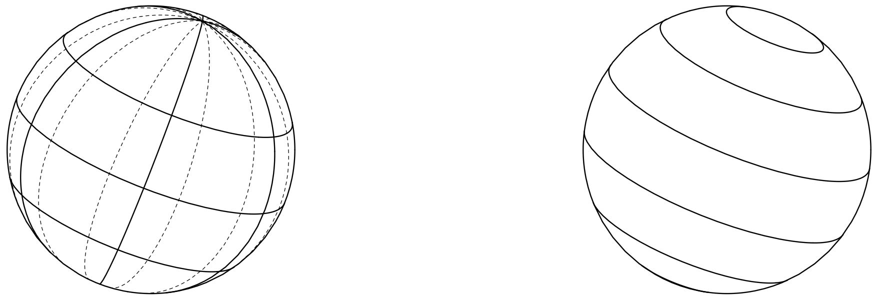


FIGURE 1: THE NODAL SETS OF u_1 (FULL LINES IN THE LEFT PICTURE), u_2 (DASHED LINES IN THE LEFT PICTURE) AND u_3 (RIGHT PICTURE)

Hence the spherical harmonic

$$u = u_1 + \varepsilon_2(u_2 + \varepsilon_1 u_3)$$

is non-singular for sufficiently small $\varepsilon_2 > 0$. By the small perturbation theorem (see e.g. [VI] or [GU1]) all negative nodal domains remain, but all positive nodal domains melt into one another and just one domain remains, i.e. $\mu^-(u) = \mu^-(u_1)$ and $\mu^+(u) = 1$ (see figure 2). Therefore

$$\mu(u) = \mu^-(u_1) + 1 = k^2 + k + 1$$

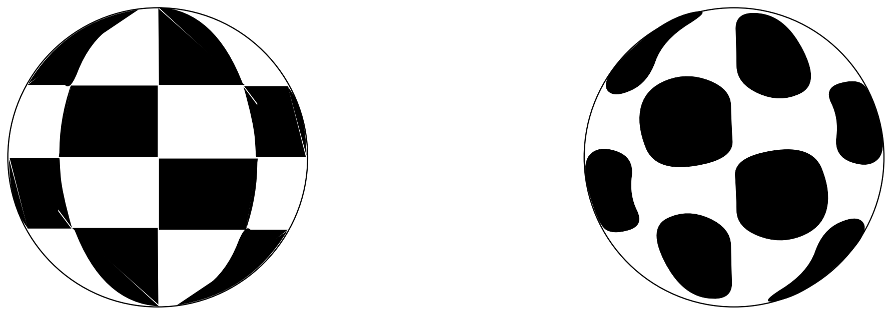


FIGURE 2: THE POSITIVE (WHITE) AND NEGATIVE (BLACK) NODAL DOMAINS OF u_1 AND $u = u_1 + \varepsilon_2(u_2 + \varepsilon_1 u_3)$.

Case (ii): $\ell = 2k-1$

We set

$$u_1 = P_\ell^k(\cos\theta)\cdot\sin(k\phi)$$
$$u_2 = P_\ell^\ell(\cos\theta)\cdot\sin(\ell\phi+\delta)$$
$$u_3 = P_\ell^0(\cos\theta)$$

The nodal set of u_1 consists of k meridians and $k-1$ latitude circles. Hence $\mu(u_1) = 2k^2$.

The nodal set of u_2 contains ℓ meridians. At a proper choice of δ, u_1 and u_2 have not a common straight line and between two neighboring meridians of u_1 there are always two meridians of u_2 with one exception where only one meridian is between them (see figure 3).

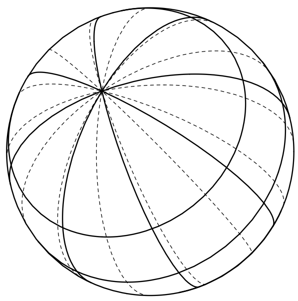


FIGURE 3: THE NODAL SETS OF u_1 AND u_2 (DASHED LINES).

Since u_3 assumes a negative value at the north-pole and a positive value at the south-pole we can split the S^2 into two hemispheres, so that the function $u_2 + \varepsilon_1 u_3$ assumes positive values at all singular points of u_1 contained in one hemisphere and negative values at all the other, for sufficiently small $\varepsilon_1 > 0$.

Hence the function

$$u = u_1 + \varepsilon_2(u_2 + \varepsilon_1 u_3)$$

is non-singular for sufficiently small $\varepsilon_2 > 0$. All negative (positive) nodal domains of u_1, which are contained in one hemisphere, remain. All positive (negative) nodal domains in this hemisphere melt into one another and into those positive (negative) nodal domains which cut the boundary of the hemisphere.

Now this boundary cuts the nodal set of u_1 in exactly $\ell = 2k - 1$ different (pairs of antipodal) points, and hence it cuts $4k - 2$ nodal domains. Because of inversion symmetry each hemisphere contains half of the other $(k-1)\cdot(k-1)$ nodal domains. Hence there are $\left[\frac{(k-1)^2+1}{2}\right]$ positive (negative) in one hemisphere. (If not u must be replaced by $-u$.) We obtain

$$\mu(u) = (k-1)^2 + 2 + [(k-1) \bmod 2]$$

as proposed.□

Remark:
After making a lot of numerical experiments we can conjecture that (i) gives the maximal number of nodal domains for non-singular spherical harmonics of even degree. If the degree is odd (ii) this might remain true at least for spherical harmonics of higher degree.

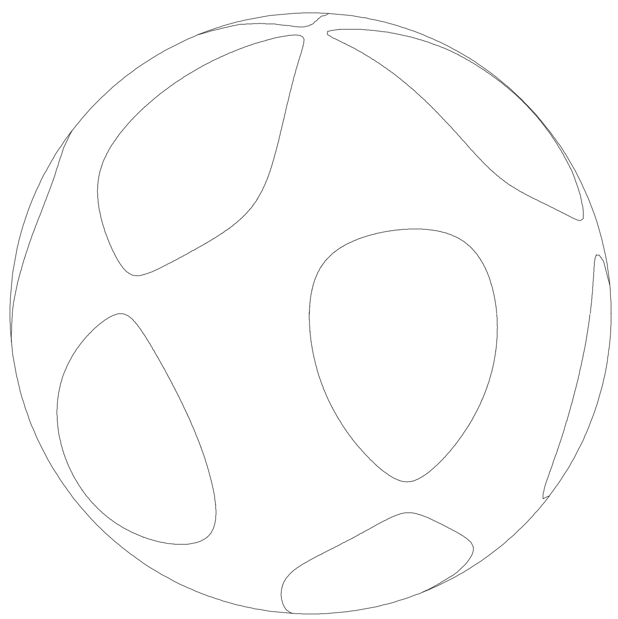


Figure 4: The nodal sets of a non-singular spherical harmonic of degree 6 with 13 nodal domains (in orthogonal projection).

The bounds

Theorem 2:
Let u be a non-singular spherical harmonic of degree $\ell \geq 2$. Then

$$\mu(u) \leq (\ell-1)(\ell-2)+3 \quad \text{for even } \ell$$
$$\mu(u) \leq (\ell-1)(\ell-3)+4 \quad \text{for odd } \ell$$

Proof:
Case of even degree:

Then there is one nodal domain ("outside"), which remains unchanged under inversion. The other domains are contained in a sector with angle $< \pi$ and are bounded by an "oval" from outside (Although they may have other nodal domains inside) (see [VI], §1.).
Now we choose the pole of our coordinate system inside of (exactly) k ovals, where $k \geq 1$. Every meridian cuts these ovals in $2k$ (pairs of) points. Therefore it cuts $\mathcal{N}(u)$ in at most $(\ell - 2k)$ more points by Bezout's theorem and hence at most $\frac{1}{2}(\ell - 2k)$ pairs of other ovals. Each of these ovals is the outer boundary of exactly one nodal domain. Hence the sectorial hull of at most $\frac{1}{2}(\ell - 2k)$ nodal domains can intersect each other at one meridian (figure 5 gives a sketch of the situation).

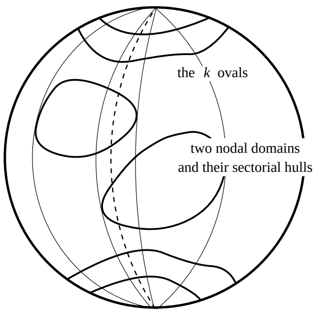


Figure 5: The situation for $\ell = 8$, where the pole is inside of $k = 2$ ovals. At most 2 sectorial hulls intersect each other at one meridian (dashed line).

By the orange slice argument the angles of these sectorial hulls are greater than $\frac{\pi}{\ell}$, and hence there are at most $(\ell - 1)$ disjoint sectorial hulls in a hemisphere. Because of inversion symmetry there are at most $2(\ell-1)$ disjoint sectorial hulls. Thus there are at most $2(\ell-1)\cdot\frac{1}{2}(\ell-2k)$ sectorial hulls of these nodal domains and so by Euler's theorem

$$\mu(u) \leq (\ell-1)(\ell-2k)+2k+1 \tag{1}$$

But since $1 \leq k \leq \frac{\ell}{2}$ we have

$$\mu(u) \leq (\ell-1)(\ell-2)+3$$

Case of odd degree:

Now there is one nodal line, which remains unchanged under inversion and which splits the sphere into two parts which contain ovals. Analogously to above each oval is the outer frontier of exactly one nodal domain with sectorial hull with angle $> \frac{\pi}{\ell}$. Again we choose the pole of our coordinate system inside of k ovals. Every meridian cuts at most $\frac{1}{2}(\ell-2k-1)$ pairs of ovals. Hence analogously $\mu(u)$ does not exceed $\frac{1}{2}(\ell-2k-1)\cdot 2(\ell-1)+2k+2$ and since $k \geq 1$:

$$\mu(u) \leq (\ell-1)(\ell-3)+4$$

as proposed.□

Remark:
As we have noticed in the chapter about algebraic curves, this estimate is equal to Harnack's theorem for the case of even degree. We are able to improve this bound a little bit. Then it is not sharp in general, but at least we can show that spherical harmonics cannot be M-curves for $\ell \geq 4$.

Theorem 2a:
Let u be a non-singular spherical harmonic of degree $\ell \geq 2$. Then

$$\mu(u) \leq (\ell-1)(\ell-3)+4$$

Corollary 1:
Let u_ℓ be a non-singular spherical harmonic of degree ℓ. Then the following sharp bounds hold.

(1) $\mu(u_1) = 2$ (3) $\mu(u_3) \leq 4$
(2) $\mu(u_2) \leq 2$ (4) $\mu(u_4) \leq 7$

Proof:

The first three cases follow immediately from Harnack's theorem, case (4) from theorem 2A. That these bounds are sharp can be seen from theorem 1.□

For the proof of theorem 2A we need the following proposition.

Lemma 1:

For every spherical harmonic u, there exists a spherical harmonic u_s of degree ℓ with at least one singular point, which has multiplicity 2, with $\mu(u) \leq \mu(u_s)$

Proof of theorem 2A:

Because of theorem 2 the case of even degree ($\ell \geq 4$) remains to show. Because of the above lemma it suffices to give an upper bound for spherical harmonics u_s with one singular point which has multiplicity 2. We can choose this singular point x_s as the pole of our coordinate system. The two tangents to u_s at this point form four sectors with angle $\frac{\pi}{2}$.
Case: The north- and south-pole are in different components of $\mathcal{N}(u_s)$.

Then there exist two simply closed curves in $\mathcal{N}(u_s)$ which contain the pole. They are the boundary of two nodal domains. These two curves cut at least two of the sectors. These two sectors are antipodal to each other and we denote their union with S', their complement with S''. Otherwise the two curves would have another common point besides the pole. Then we make two curves from the old ones which satisfy this condition.

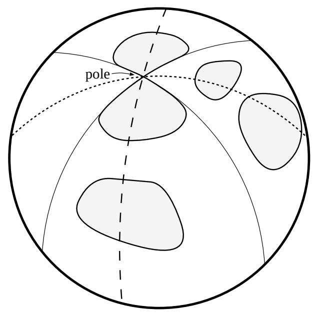


Figure 6: The situation for $\ell = 6$, $k = 0$. The dashed meridian intersects at most one, the dotted meridian at most two ovals.

Since the tangents of the curves at the pole form the boundary of S', every meridian in S' cuts both curves. Now there might be k components of u_s, which intersect every meridian (the pole is inside of these ovals). By Bezout's theorem every meridian in S' cuts $\mathcal{N}(u_s)$ in at most $(\ell - 4 - k)$ points on further components, every meridian in S'' in at most $(\ell - 2 - k)$ points on further components (see figure 6).

Analogously to the proof of theorem 2 at most $\frac{1}{2}(\ell - 4 - k)$ sectorial hulls of nodal domains of u_s with angle at the pole can intersect each other at one meridian in S', whereas in S'' there are at most $\frac{1}{2}(\ell - 2 - k)$. Hence by the orange slice argument there are at most $(\ell - 2)$ ovals, which are limited to S' and at most $2(\ell - 1)\frac{1}{2}(\ell - 4 - k)$ further ovals which are neither limited to S' nor to S'' (figure 6 sketches the situation: one of the two ovals in the one sector might cut the neighboring sector too. Both cannot). Since we have beside these ovals the k ovals which contain the pole in their inside and the two components of u_s that contain the (pair of) poles (the only singular point of u_s) we have by Euler's theorem $\mu(u_s) \leq (\ell - 1)(\ell - 4 - k) + (\ell - 2) + k + 2 + 2 + 1$ and so

$$\mu(u_s) \leq (\ell - 1)(\ell - 3) + 4$$

Case: The north- and south-pole are in the invariant component of u_s.

Since by assumption ℓ is even there must be two curves which remain unchanged under inversion and which intersect at the pole. The union of these two curves forms the invariant component of u_s. The two curves cut out four "deformed sectors" $\widetilde{S}$ from the sphere. Now either all these domains $\widetilde{S}$ contain no oval, then obviously $\mu(u_s) = 4$ and we are done. Or at least two of them contain other ovals. By a sufficiently small change of the parameters of u_s all these ovals remain but the singular points disappear. The invariant component splits into a pair of antipodal ovals such that all the other ovals are inside. We have got a non-singular spherical harmonic u'_s with $\mu(u'_s) = \mu(u_s) + 1$. For this we can use (1) from the proof for theorem 2. Since we know that we can locate the pole of our coordinate system inside of at least 2 ovals, i.e. $k \geq 2$ in the formula (1), we have

$$\mu(u'_s) \leq (\ell - 1)(\ell - 4) + 5$$

Since by assumption $\ell \geq 4$ we are done.□

Proof of lemma 1:
Let u' be any non-singular spherical harmonic of degree ℓ with $\mu(u') \neq \mu(u)$. Such an u' always exists since there are spherical harmonics of arbitrary degree with 2 or 3 nodal domains (see [LE]) on the one hand and the spherical harmonics of theorem 1 on the other. Then $u_\varepsilon = (1-\varepsilon)u + \varepsilon u'$ is a spherical harmonic of degree ℓ. By a continuity argument the isotopy types of u and u_ε are the same for sufficiently small $\varepsilon > 0$. It can change for increasing ε if and only if singular points occur. Since no nodal domain can shrink until it disappears (this follows from the Faber-Krahn theorem or from Harnack's inequality) this happens if and only if k (local) branches join at one point ($k \geq 2$). Then there arises a singular point of multiplicity k. If all these branches correspond to different components of u, then the number of components of the nodal sets decreases about k. If some of these branches correspond to the same component this number decreases less than k. We have for a proper $\varepsilon > 0$

$$\nu(u_\varepsilon) = \nu(u) + k \quad \text{and} \quad \zeta(u_\varepsilon) \geq \zeta(u) - k$$

Therefore $\mu(u_\varepsilon) \geq \mu(u)$. and we can choose u_ε for our u_s.□

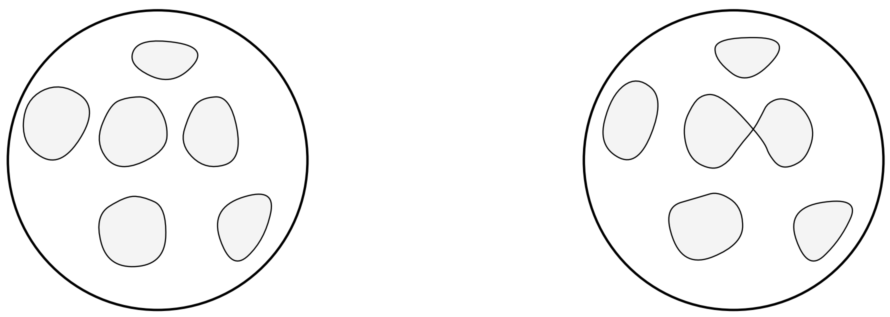


Figure 7: u and u_s

Er glaubte nämlich, die Erkenntnis jeder Kleinigkeit, also zum Beispiel auch eines sich drehenden Kreisels, genüge zur Erkenntnis des Allgemeinen. Darum beschäftigte er sich nicht mit den großen Problemen, das schien ihm unökonomisch. War die kleinste Kleinigkeit wirklich erkannt, dann war alles erkannt, deshalb beschäftigte er sich nur mit dem sich drehenden Kreisel. Und immer wenn die Vorbereitungen zum Drehen des Kreisels gemacht wurden, hatte er Hoffnung, nun werde es gelingen, und drehte sich der Kreisel, wurde ihm im atemlosen Laufen nach ihm die Hoffnung zur Gewißheit, hielt er aber dann das dumme Holzstück in der Hand, wurde ihm übel [. . .]

Franz Kafka
in: Der Kreisel

Nodal pattern

In the following we want to use "graphs" to describe "nodal patterns". All the definitions and results on graphs are taken from Mayeda's book about graph theory [MA]. We use this concept to write down the situation on the separate nodal domains on the sphere. Moreover we are able to code a few results with this concept.

Graphs

We associate to every spherical harmonic[1] u a graph $\mathcal{G}(u)$ in the following way:

The set of *vertices* $\omega(u)$ are the nodal domains of u. The set of *edges* $\mathcal{E}(u) \subset \omega \times \omega$ are those pairs (v, v') of nodal domains (vertices), which are connected via a common nodal line, i.e. when $\overline{v} \cup \overline{v}'$ is locally diffeomorphic to the real line except (maybe) for a finite number of points.

To each of these graphs there exists a *representation* on the sphere, where each vertex v is represented by a point and each edge (v, v') by an injective curve connecting these points. Figure 1 gives an example of the nodal lines of a spherical harmonic, its Graph and a representation of this.

Since we can imagine the nodal lines as the representation of a graph, the definition $\mathcal{G}(u)$ is similar to that of the dual graph of the nodal lines.

Before we continue: We say two graphs are *the same*, if they are graphs of spherical harmonics u_1, u_2 of the same degree and if there exists a bijective map $\beta\colon \omega(u_1) \to \omega(u_2)$ where $(v, v') \in \mathcal{E}(u_1) \iff (\beta(v), \beta(v')) \in \mathcal{E}(u_2)$.

A few elementary properties of these graphs are given by

Lemma 1:
Let u be a spherical harmonic. Then
(1) $\mathcal{G}(u)$ is planar and connected and does not contain any self-loops
(2) Each circuit in $\mathcal{G}(u)$ has an even number of vertices (it is bipartite).

[1] The whole concept and all results hold if we have a spherical polynomial without multiple components with the property that each of its singular points of multiplicity m there are m distinct tangents.

Proof:

The first statement is rather trivial. The second follows from Harnack's inequality (see chapter about spherical harmonics).□

Much more information we get from the following proposition.

Lemma 2:

If u has no singular points, then $\mathcal{G}(u)$ is a tree.

Proof:

We show the equivalent proposition: If $\mathcal{G}(u)$ contains a circuit, then u possesses a singular point.

Let $\mathcal{G}'$ be a subgraph of $\mathcal{G}(u)$ which is a circuit with k vertices $v_1, \dots v_k$. Then there is a representation of $\mathcal{G}'$ which is a simply closed curve τ in $\overline{v_1 \cup \ldots \cup v_k} \subseteq S^2$ which intersects all these nodal domains and which cuts $\mathcal{N}(u)$ in exactly k points. None of these points is a singular point of u (otherwise the proposition is shown).
The nodal set of u consists either of injective curves, which connect singular points of u, or of simply closed curves which is contained in the frontier of two nodal domains (see [CS]). Now let γ be such a curve in $\mathcal{N}(u)$, which is cut by τ. Then γ cannot be a closed curve and hence u has a singular point as proposed.
Otherwise τ would go from a nodal domain v_j via γ to a nodal domain v_l and from there again back to v_j. Hence $\mathcal{G}'$ would not be a circuit, a contradiction.□

Next we need the concept of an "antipodal map". We call a map $c\colon \omega(u) \to \omega(u)$ *antipodal*, or c is an *antipodal map*, if $c(v) = \{x \in S^2 \colon -x \in v\}$, i.e. the antipodal nodal domain of v.

Obviously we have

Lemma 3:

$(v, v') \in \mathcal{E}(u) \iff (c(v), c(v')) \in \mathcal{E}(u)$.

Lemma 4:

(1) If ℓ is odd, then $v \neq c(v)\ \forall v \in \omega$.

(2) If ℓ is even, then there is at most one vertex $v_0 \in \omega(u)$ with $c(v_0) = v_0$. If we delete this vertex then the graph disconnects into several pairwise equal subgraphs.

Proof:

(1) follows immediately from inversion anti-symmetry.

(2) Assume v_0 exists. Then there is a simply closed curve in v_0 which remains unchanged under inversion and which cuts out two equal connected subsets of the sphere ("deformed hemispheres"). Hence there cannot be another such vertex and the nodal domains $v \neq v_0$, $c(V)$ must be in different subsets. From this and lemma 3 the proposition follows.□

The next lemmata can be used to find antipodal vertices in a graph. Before we do this we define a *path* between two vertices v, v' in $\mathcal{G}(u)$ is a sequence of edges $\Gamma(v, v') = \{(v_0 = v, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n = v')\}$ of the graph. $\Gamma(v, v)$ might be the empty set.
The *length* of the path is $|\Gamma(v, v')|$, i.e. the number of edges. In the following we will write "$v_i \in \Gamma(v, v')$" for short, if we mean that there exists a vertex v_j so that $(v_i, v_j) \in \Gamma(v, v')$.
The *distance* between two vertices v, v' in a graph is the minimal length of a path $\Gamma(v, v')$, i.e. $d(v, v') = \min\limits_{\Gamma(v,v')} |\Gamma(v, v')|$ ($d(.,.)$ is a metric on $\mathcal{G}$.)
The *diameter* $d(\mathcal{G})$ of a graph is the maximal distance between two vertices $d(\mathcal{G}) = \max\limits_{v,v' \in \omega} d(v, v')$.

With this notation we can show

Theorem 1:

$$d(\mathcal{G}(u)) = \text{even} \iff \ell = \text{even}$$

Theorem 2:

$$d(\mathcal{G}(u)) \leq \ell$$

This theorem can be improved in the following way:

Theorem 2A:

For any two vertices $v, w \in \omega(u)$ there is a path $\Gamma(v, c(v))$ of length $|\Gamma| \leq \ell$.

For the proof of these theorems we need two lemmata.

Lemma 4:

Let $v, w \in \omega(u)$. Then the paths $\Gamma(v, c(v))$ and $\Gamma(w, c(w))$ or $c\left(\Gamma(v, c(v))\right)$ and $\Gamma(w, c(w))$ have a common vertex. ($c(\Gamma)$ is the antipodal path to Γ.)

Proof:

Suppose neither w nor $c(w)$ is a vertex of $\Gamma(v, c(v))$ or $c(\Gamma(v, c(v)))$ (otherwise nothing remains to show). Without loss $w \neq c(w)$. Otherwise change the roles of v and w (cf. lemma 4(2)). Let $g \subseteq \omega(u)$ be the set of vertices of $\Gamma(v, c(v)) \cup c(\Gamma(v, c(v)))$ and $K = \bigcup_{v' \in g} v'$. $\overline{K}$ is connected and contains a curve which remains unchanged under inversion. By assumption $S^2 \setminus \overline{K} \neq \emptyset$ and hence decomposes in at least 2 components which contain w and $c(w)$, respectively. Therefore any path $\Gamma(w, c(w))$ must contain a vertex from g.□

Lemma 5:

Suppose $\omega_v = \{v' \in \omega(u) \colon d(v, v') = d(\mathcal{G}(u))\}$ is not empty. Then $c(v) \in \omega_v$.

Proof:

Indirect. Suppose there exist two vertices $v, w \in \omega(u)$ with $d(v, w) = d := d(\mathcal{G}(u))$ and $d(v, c(v)) < d$. Let $\Gamma_1 = \Gamma(v, c(v))$ with $|\Gamma_1| = d(v, c(v))$ and $\Gamma_2 = \Gamma(w, c(w))$ with $|\Gamma_2| = d(w, c(w))$ paths.

By lemma 4 Γ_1 and (without loss) γ_2 have a common vertex v_0. From the triangle inequality we have $d = d(v, w) \leq d(v, v_0) + d(v_0, w)$ and hence $d(w, v_0) \geq d - d(v, v_0)$. Equally $d(c(w), v_0) \geq d - d(c(v), v_0)$. Since $v_0 \in \Gamma_1$ and $\Gamma_1 = d(v, c(v))$ we have $d(v, v_0) + d(v_0, c(v)) = d(v, c(v))$ and therefore

$$
\begin{aligned}
d(w, c(w)) &= d(w, v_0) + d(v_0, c(v)) \geq (d - d(v, v_0)) + (d - d(c(v), v_0)) \\
&= 2d - (d(v, v_0) + d(v_0, c(v))) = 2d - \underbrace{d(v, c(v))}_{<d} \\
&> 2d - d = d
\end{aligned}
$$

a contradiction.□

Proof of theorem 1:

From Harnack's inequality we know that u must change sign at $\mathcal{N}(u)$. Therefore we obtain from inversion symmetry

$$
|\Gamma(v, c(v))| = \text{even} \qquad \Longleftrightarrow \qquad \ell = \text{even}
$$

Hence the result follows from lemma 5.□

Proof of theorem 2A:

Let L be a line through a point $x \in v$ and $y \in w$. Then L cuts v, $c(v)$ and w. Without loss we can assume that L does not contain any singular point of u, since there are at most finitely many. Because of Bezout's theorem L cuts $\mathcal{N}(u)$ in at most ℓ points. Hence it cuts at most $\ell + 1$ nodal domains of u. By construction L can be seen as a part of a representation of the graph $\mathcal{G}(u)$ on the sphere, hence as the representation of a path from v to $c(v)$ via w. Hence there is a path of length $|\Gamma| \leq \ell$ as proposed.□

Proof of theorem 2:
This either follows from theorem 2a, or we use lemma 5. From this we can conclude that there is a vertex $v \in \omega(u)$ with $d(v, c(v)) = d(\mathcal{G}(u))$. Then there is a line L through a point $x \in v$ (and hence through $-x \in c(v)$), which does not contain any singular point of u. Analogously to the above proof there is a path from v to $c(v)$ with length less than or equal to ℓ.□

Obviously we can deduce a lot of other simple "theorems". For example

Proposition 1:
There is at most one $v_0 \in \omega(u)$ with $d(v_0, c(v_0)) = 1$ (except $c(v_0)$).

Proof:
Follows immediately from lemma 4.

Or we can code a few results with this concept. For example

Proposition 2:
Let $v \in \omega(u)$ be a vertex with $d(v, c(v)) = \ell - 2$. Then there are at most 2ℓ vertices of degree 1.□

Proof:
Follows analogously to the proof of theorem 2 about non-singular curves.□

All in all it leads to something which someone once upon a time has called *"Micky Mouse Mathematics"*. But to continue on this path is very hard. One gets more and more (complicated) proposition, which limit the possible graphs. But these restrictions are too weak to derive a suitable bound.

The Eigenspace

In this chapter we want to deal with the eigenspace of the Laplace-Beltrami operator on the sphere.

Denote $\tilde{E}_\ell$ the eigenspace to the eigenvalue $\lambda = \ell(\ell-1)$. We already know that this space is of dimension $2\ell - 1$ and hence isomorphic to $\mathbb{R}^{2\ell-1}$. Since the nodal lines do not change when we multiply an eigenfunction with a non-zero constant, we are just interested in the normalized function. We denote the set of these functions by

$$E_\ell := \{u \in \tilde{E}_\ell : \|u\|_2 = 1\} \cong S^{2\ell}$$

We also know that the Laplace-Beltrami operator is invariant under rotation. Hence there is a group $\mathcal{R}$ acting on the eigenspace E_ℓ which just rotates the eigenfunctions on the sphere. This group is isomorphic to $SO(3)$. Now if we identify all the functions which can be transformed into each other by a simple revolution we get a new space $E_\ell/\mathcal{R}$, the space of the orbits of the $u \in E_\ell$ under the action of $\mathcal{R}$. The resulting space is called an *orbifold* (see [SI] or [GW],§1). All orbits are of dimension 3 except the orbit of zonal harmonics (see [SW], p. 143ff), which is of dimension 2. Hence this space is not a Riemannian Manifold and we do not deal with it because it would lead to too structural questions for which we had to provide a lot of new tools.

Another possible study of the eigenspace is to split it up into "cells" by using the number of singular points of the spherical harmonics u. We define

$$E_\ell^k(x) := \{u \in E_\ell : \nu_u(x) \geq k\}$$

and

$$E_\ell^k := \bigcup_{x \in S^2} E_\ell^k(x)$$

With a decomposition like this, we obtain a CW-decomposition of the eigenspace. It is related to the concepts of *roughness* and *degree of non-roughness* (see [GU1], p. 6), which has been developed by A. A. Andronov and L. S. Pontryagin for a special class of differential equations[1].

In this paper we are much more modest. We just study the structure of $E_\ell \setminus E_\ell^1$. We want to find some results on the size of its components.

[1] D. A. Gudkov applies this concept for plane projective algebraic curves in [GU2].

The distance function

We want do find a function on the eigenspace which gives us for each spherical harmonic u the (euclidean) distance from E_ℓ^1. This function obviously is given by

$$d(u) := \inf_{x \in S^2} d(u, x)$$

where

$$d(u, x) := \inf\{\|u - u'\|_2 \colon u' \in E_\ell^1(x)\}$$

To calculate the values of this function, we have to study the "space of coefficients", i.e. the space of all coefficients of spherical harmonics with respect to a given ON-base.

So let $\{u_0, \dots u_{2\ell}\}$ be any set of orthogonal spherical harmonics of degree ℓ. Then obviously this set is an ON-base for the eigenspace (to the eigenvalue $\ell(\ell-1)$). Each spherical harmonic u can be uniquely determined in terms of this functions, i.e. $u(x) = \sum_{k=0}^{2\ell} c_u^k u_k(x)$, for some uniquely determined c_u^k. We can abbreviate this by putting the coefficients of u into a vector $\vec{c}_u \in \mathrm{I\!R}^{2\ell+1}$ and the values of the base functions at a point x into a vector $\vec{u}(x)$. The spherical harmonic u then can be written as the scalar product of these.
For later use it will be convenient not to use this ON-base itself but

$$\vec{u}(x) := \sqrt{\frac{4\pi}{2\ell+1}}\,(u_0(x), \dots, u_{2\ell}(x))$$

Any spherical harmonic can be written as

$$u(x) = \eta \langle \vec{c}_u, \vec{u}(x) \rangle$$

where

$$\eta = \sqrt{\frac{2\ell+1}{4\pi}}$$

Since there is a canonical isomorphism between the eigenspace E_ℓ and the space of coefficients we will denote both with the symbol E_ℓ (or E_ℓ^1,respectively).

Remark:

As can easily be seen, $u(x) = 0$ if and only $\vec{c}_u$ is perpendicular to $\vec{u}(x)$.
If we choose $\vec{c}_u = \vec{u}(y)$ then

$$u(x) := \eta\langle\vec{c}_u, \vec{u}(y)\rangle = \eta\langle\vec{u}(x), \vec{u}(y)\rangle = \eta P_\ell(x \cdot y)$$

is a zonal harmonic (see [SW], p. 143 ff and [MC], p. 7 ff).
u takes its maximum at y and $u(y) = \eta|\vec{u}(y)|^2 = \eta$. Hence η is the maximum of all normalized spherical harmonics with eigenvalue $\lambda = \ell(\ell + 1)$, i.e.

$$\eta = \sup_{\|u\|=1} \sup_{x \in S^2} |u(x)|^2$$

Next we differentiate this vector-valued function with respect to the spherical coordinates and define

$$\vec{v}(x) := \sqrt{\frac{2}{\ell(\ell-1)}} \frac{\partial}{\partial\theta}\vec{u}(x)$$

$$\vec{w}(x) := \sqrt{\frac{2}{\ell(\ell-1)}} \frac{1}{\sin\theta} \frac{\partial}{\partial\phi}\vec{u}(x)$$

It is obvious that $u \in E_\ell^1$ if and only if $u(x) = 0$ and $|\nabla u(x)| = 0$, i.e. $\frac{\partial}{\partial\theta}u(x) = \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}u(x) = 0$. Therefore we have

$$E_\ell^1(x) = \{u \in E_\ell \colon \langle\vec{c}_u, \vec{u}(x)\rangle = \langle\vec{c}_u, \vec{v}(x)\rangle = \langle\vec{c}_u, \vec{w}(x)\rangle = 0\}$$

and hence the set of spherical harmonics having a singular point at x is a linear subspace of codimension 3 of the eigenspace. The distance of u from this set is then given by the length of the projection of the coefficient c_u onto the orthogonal complement $(E_\ell^1(x))^\perp$, which is the linear span of $\vec{u}(x)$, $\vec{v}(x)$ and $\vec{w}(x)$. Luckily these three vector-valued functions have a nice property, which can be deduced from the addition theorem for spherical harmonics.

Proposition 1:

$\{\vec{u}(x), \vec{v}(x), \vec{w}(x)\}$ is an ON-system in $\mathbb{R}^{2\ell+1}$ for all $x \in S^2 \setminus \{\text{pole}\}$.

From this we immediately have

Lemma 1:

$$d^2(u,x) = \langle \vec{c}_u, \vec{u}(x)\rangle^2 + \langle \vec{c}_u, \vec{v}(x)\rangle^2 + \langle \vec{c}_u, \vec{w}(x)\rangle^2$$

Proof:
Because of proposition 1, $\langle \vec{c}_u, \vec{u}(x)\rangle \vec{u}(x) + \langle \vec{c}_u, \vec{v}(x)\rangle \vec{v}(x) + \langle \vec{c}_u, \vec{w}(x)\rangle \vec{w}(x)$ is the projection of $\vec{c}_u$ onto $(E^1_\ell(x))^\perp$ and the result follows.□

Now we already know that

$$\langle \vec{c}_u, \vec{u}(x)\rangle^2 = (\frac{1}{\eta} u(x))^2$$

and analogously

$$\langle \vec{c}_u, \vec{v}(x)\rangle^2 = \left(\frac{1}{\eta}\frac{2}{\lambda}\frac{\partial u}{\partial \theta}(x)\right)^2$$

$$\langle \vec{c}_u, \vec{w}(x)\rangle^2 = \left(\frac{1}{\eta}\frac{2}{\lambda}\frac{1}{\sin\phi}\frac{\partial u}{\partial \theta}(x)\right)^2$$

where $\lambda = \ell(\ell-1)$. Thus we have from lemma 1

Proposition 2:

$$d^2(u) = \min_{x\in S^2} \frac{1}{\eta^2}\left(u^2(x) + \frac{2}{\lambda}|\nabla u(x)|^2\right)$$

We can show an upper bound for this distance function.

Theorem 1:

$$d(u) \le \sqrt{\frac{1}{2\ell+1}(2+\frac{1}{\lambda-1})}\,\|u\|_2$$

The following theorem shows, that this bound is of the right order.

Theorem 2:
Let u be a normalized zonal harmonic with eigenvalue $\lambda = \ell(\ell+1)$. Then

$$\tfrac{2}{\pi\ell}\exp\left(-\tfrac{2}{3\ell}\right) < d^2(u) < \tfrac{2}{\pi\ell}\exp\left(\tfrac{1}{6\ell}\right) \qquad \text{for even } \ell$$

$$\frac{2}{\ell\pi}\exp(-\frac{2}{3(\ell+1)}) < d^2(u\,) < (\frac{2}{\ell\pi} + \frac{2}{\pi}\frac{1}{\ell^3+\ell^2-\ell})\exp(\frac{1}{6(\ell+1)}) \qquad \text{for odd } \ell$$

Corollary:
Let u be a normalized zonal harmonic. Then

$$d^2(u) = \frac{2}{\pi\ell} + O\left(\frac{1}{\ell^2}\right)$$

The proofs of proposition 1 and of the theorems are given below.

We want to find a (positive) lower bound for the least local maximum of the distance function. If we had found one, we can start to go through the eigenspace for all possible non-singular spherical harmonics (except for homeomorphisms) for a given ℓ by means of computer graphics. The lower bound on the distance function would ensure us, that we find all. Numerical experiments (for example the trials to plot the non-singular curves of theorem 1 about non-singular curves for higher ℓ; see figure 4 in that chapter) show us, that this bound decreases very rapidly if ℓ increases. Indeed we can non derive such a lower bound yet.

Proof of proposition 1

We will prove this proposition by use of a generalization of the addition theorem for spherical harmonics (see [MC], p. 10).

Let $x \in S^2 \setminus \{\text{pole}\} \mapsto \{\vec{\alpha}_1(x), \vec{\alpha}_2(x)\}$ be a continuous function, where for all x $\vec{\alpha}_1(x), \vec{\alpha}_2(x)$ are two orthonormal vectors in the tangent plane on the sphere at x. Let $U_0(x), \ldots, U_{2\ell}(x)$ denote the harmonic polynomials corresponding to the base functions $u_0(x), \ldots, u_{2\ell}(x)$. Then we define functions on the sphere by taking the partial derivative with respect to $\vec{\alpha}_1(x)$ and $\vec{\alpha}_2(x)$.

$$v_{ij}^m(x) := \left(\frac{\partial}{\partial \vec{\alpha}_1(x)}\right)^i \left(\frac{\partial}{\partial \vec{\alpha}_2(x)}\right)^j U_m(x)$$

Then we can show

Lemma 2: (Addition theorem)

$$\sum_{m=0}^{2\ell} v_{i_1 j_1}^m(x) \cdot v_{i_2 j_2}^m(x) = \text{const} \qquad \text{for all } i_1, j_1, i_2, j_2 \in \mathbb{N}_0$$

independent of the choice of the ON-base $\{u_m\}$ and of the $\vec{\alpha}_1(x), \vec{\alpha}_2(x)$.

Proof:
First we show: $\sum\limits_{m=0}^{2\ell} v_{i_1 j_1}^m(x) \cdot v_{i_2 j_2}^m(x)$ is independent of the choice of the ON-base $\{u_m\}$.
Let $\{u_0, \ldots, u_{2\ell}\}$ and $\{u_0', \ldots, u_{2\ell}'\}$ be two arbitrary ON-bases for the space of spherical harmonics with eigenvalue $\lambda = \ell(\ell+1)$. Then

$$u_k(x) = \sum_{m=0}^{2\ell} a_{km} u_m'(x)$$

Now

$$\delta_{nk} = \langle u_n, u_k \rangle = \sum_{m=0}^{2\ell} \sum_{i=0}^{2\ell} a_{nm} a_{ki} \underbrace{\langle u_m', u_i' \rangle}_{=\delta_{mi}} = \sum_{m=0}^{2\ell} a_{nm} a_{km}$$

i.e. for $A = (a_{km})$, $A^t \cdot A = I$ and thus $A \cdot A^t = I$ and so

$$\sum_{m=0}^{2\ell} a_{mn} a_{mk} = \delta_{nk}$$

Therefore it follows that

$$\sum_{m=0}^{2\ell} v_{i_1 j_1}^m(x) \cdot v_{i_2 j_2}^m(x) =$$

$$= \sum_{k=0}^{2\ell} \left(\left(\frac{\partial}{\partial \vec{\alpha}_1(x)} \right)^{i_1} \left(\frac{\partial}{\partial \vec{\alpha}_2(x)} \right)^{j_1} \underbrace{U_k}_{\sum\limits_{m=0}^{2\ell} a_{km} U_m'} \right)\Big|_x \cdot$$

$$\cdot \left(\left(\frac{\partial}{\partial \vec{\alpha}_1(x)} \right)^{i_2} \left(\frac{\partial}{\partial \vec{\alpha}_2(x)} \right)^{j_2} \underbrace{U_k}_{\sum\limits_{n=0}^{2\ell} a_{kn} U_n'} \right)\Big|_x =$$

$$= \sum_{k=0}^{2\ell} \left(\sum_{m=0}^{2\ell} a_{km} \underbrace{\left(\frac{\partial}{\partial \vec{\alpha}_1(x)} \right)^{i_1} \left(\frac{\partial}{\partial \vec{\alpha}_2(x)} \right)^{j_1} U_m'\Big|_x}_{=v'^m_{i_1 j_1}(x)} \right) \cdot$$

$$\cdot \left(\sum_{n=0}^{2\ell} a_{kn} \underbrace{\left(\frac{\partial}{\partial \vec{\alpha}_1(x)} \right)^{i_2} \left(\frac{\partial}{\partial \vec{\alpha}_2(x)} \right)^{j_2} U_n'\Big|_x}_{=v'^m_{i_2 j_2}(x)} \right) =$$

$$= \sum_{m=0}^{2\ell} \sum_{n=0}^{2\ell} v'^m_{i_1 j_1}(x) \cdot v'^n_{i_2 j_2}(x) \underbrace{\sum_{k=0}^{2\ell} a_{km} a_{kn}}_{=\delta_{nm}} =$$

$$= \sum_{m=0}^{2\ell} v'^m_{i_1 j_1}(x) \cdot v'^m_{i_2 j_2}(x)$$

Now let x and y be two points on the sphere and A an orthogonal transform such that $Ay = x$. Then we obtain from the ON-base $\{u_m\}$ a new base $\{u'_m(x) = u_m(Ax)\}$. Then we have

$$\sum_{m=0}^{2\ell} v^m_{i_1 j_1}(y) \cdot v^m_{i_2 j_2}(y) = \sum_{m=0}^{2\ell} v'^m_{i_1 j_1}(Ay) \cdot v'^m_{i_2 j_2}(Ay) =$$
$$= \sum_{m=0}^{2\ell} v'^m_{i_1 j_1}(x) \cdot v'^m_{i_2 j_2}(x) =$$
$$= \sum_{m=0}^{2\ell} v^m_{i_1 j_1}(x) \cdot v^m_{i_2 j_2}(x)$$

i.e. this sum is independent from x, it is a constant.

In the same way (by using orthogonal transformations that do not change a given point x) we can show that this sum is independent of the choice of the vectors $\alpha_i(x)$, as proposed.□

The sum in the above lemma has a symmetric property.

Lemma 3:

(1) $\sum_{m=0}^{2\ell} v^m_{i_1 j_1}(x) \cdot v^m_{i_2 j_2}(x) = (-1)^{i_1+i_2} \sum_{m=0}^{2\ell} v^m_{j_1 i_1}(x) \cdot v^m_{j_2 i_2}(x)$

(2) $\sum_{m=0}^{2\ell} v^m_{i_1 j_1}(x) \cdot v^m_{i_2 j_2}(x) = 0 \qquad \text{if } i_1 + i_2 + j_1 + j_2 = \text{ odd}$

Proof:

(1) Let $v^m_{i_1,j_1}(x), v^m_{i_2,j_2}(x)$ be the derivatives according to $\vec{\alpha}_1(x), \vec{\alpha}_2(x)$ and $v'^m_{i_1,j_1}(x)$, $v'^m_{i_2,j_2}(x)$ those according to $\vec{\alpha}'_1(x) = \vec{\alpha}_2(x), \vec{\alpha}'_2(x) = -\vec{\alpha}_1(x)$, where $-\vec{\alpha}_1(x)$ denotes the vector in the tangent plane at x with the same base point x with orientation opposite to $\vec{\alpha}_1(x)$.

Then $v'^m_{j_1,i_1}(x) = (-1)^{i_1} v^m_{i_1,j_1}(x)$ and $v'^m_{j_2,i_2}(x) = (-1)^{i_2} v^m_{i_2,j_2}(x)$. On the other hand we get $\vec{\alpha}'_1(x), \vec{\alpha}'_2(x)$ by rotating $\vec{\alpha}_1(x), \vec{\alpha}_2(x)$ about x. Thus it follows from lemma 2

$$\sum_{m=0}^{2\ell} v^m_{i_1 j_1}(x) \cdot v^m_{i_2 j_2}(x) = \sum_{m=0}^{2\ell} (-1)^{i_1} v'^m_{j_1 i_1}(x) \cdot (-1)^{i_2} v'^m_{i_2 j_2}(x)$$
$$= (-1)^{i_1+i_2} \sum_{m=0}^{2\ell} v'^m_{j_1 i_1}(x) \cdot v'^m_{i_2 j_2}(x)$$
$$= (-1)^{i_1+i_2} \sum_{m=0}^{2\ell} v^m_{j_1 i_1}(x) \cdot v^m_{i_2 j_2}(x)$$

(2) From (1) we have

$$\sum_{m=0}^{2\ell} v^m_{i_1 j_1}(x) \cdot v^m_{i_2 j_2}(x) = (-1)^{i_1+i_2}(-1)^{j_1+j_2} \sum_{m=0}^{2\ell} v^m_{i_1 j_1}(x) \cdot v^m_{i_2 j_2}(x)$$

and the result follows.□

Now we are ready to prove proposition 1.

First we notice, that $\frac{\partial}{\partial\theta} u_m(x)$ and $\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} u_m(x)$ can be treated as $v^m_{10}(x)$ and $v^m_{01}(x)$, respectively. For this we choose $\vec{\alpha}_1(x) = \vec{\alpha}_\theta(x)$ and $\vec{\alpha}_2(x) = \vec{\alpha}_\phi(x)$, respectively, which are the tangent vectors of the meridians and latitude circles.

Unit length:

From lemma 2 we know, that $\|\vec{u}(x)\|$, $\|\vec{v}(x)\|$ and $\|\vec{w}(x)\|$ do not depend on $x \in S^2$. By the addition theorem ([MC], p. 10) $\sum\limits_{k=0}^{2\ell} |u_k(x)|^2 = \frac{2\ell+1}{4\pi}$, hence $\|\vec{u}(x)\| = 1$.

For estimating $\|\vec{v}(x)\|$ and $\|\vec{w}(x)\|$ we calculate

$$\begin{aligned} 0 &= \frac{1}{2}\Delta \underbrace{\sum_{k=0}^{2\ell} \big(u_k(x)\big)^2}_{=\text{const.}} = \frac{1}{2}\sum_{k=0}^{2\ell} \Delta\big(u_k^2(x)\big) \\ &= \sum_{k=0}^{2\ell} |\nabla u_k|^2 + \sum_{k=0}^{2\ell} \underbrace{u_k \Delta u_k}_{=-\lambda u_k^2} \\ &= \sum_{k=0}^{2\ell} \Big(\big(\frac{\partial}{\partial\theta} u_k\big)^2 + \big(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} u_k\big)^2 \Big) - \underbrace{\lambda \sum_{k=0}^{2\ell} u_k^2}_{=\lambda\|\vec{u}(x)\|^2=\lambda} \end{aligned}$$

Thus we have

$$\sum_{k=0}^{2\ell} \big(\frac{\partial}{\partial\theta} u_k\big)^2 + \sum_{k=0}^{2\ell} \big(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} u_k\big)^2 = \lambda$$

Since by lemma 3

$$\sum_{k=0}^{2\ell} \big(\frac{\partial}{\partial\theta} u_k\big)^2 = \sum_{k=0}^{2\ell} \big(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} u_k\big)^2$$

$\|\vec{v}(x)\| = \|\vec{w}(x)\| = 1$ follows.

Orthogonality:

From Lemma 3 we immediately have

$$\langle \vec{u}(x), \vec{v}(x) \rangle = \langle \vec{u}(x), \vec{w}(x) \rangle = 0$$

For calculating $\langle \vec{v}(x), \vec{w}(x) \rangle$ we notice, that by lemma 2, $\sum_{k=0}^{2\ell} \left(\frac{\partial}{\partial \vec{\alpha}} u_k(x)\right)^2$ does not depend on $\vec{\alpha}$ but on $|\vec{\alpha}|$.

Since $\left(\frac{\partial}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right) u_k(x) = \binom{1}{1} \cdot \text{grad}\ \ u(x)$ and $\left|\binom{1}{1}\right|^2 = 2$, we therefore have

$$\sum_{k=0}^{2\ell} \left((\frac{\partial}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial \phi}) u_k(x) \right)^2 = 2 \sum_{k=0}^{2\ell} \left(\frac{\partial}{\partial \theta} u_k(x) \right)^2 = \lambda$$

On the other hand

$$\begin{aligned} &\sum_{k=0}^{2\ell} \left(\frac{\partial}{\partial \theta} u_k(x) + \frac{1}{\sin\theta}\frac{\partial}{\partial \phi} u_k(x) \right)^2 \\ &= \underbrace{\sum_{k=0}^{2\ell} \left(\frac{\partial}{\partial \theta} u_k(x) \right)^2 + \sum_{k=0}^{2\ell} \left(\frac{1}{\sin\theta}\frac{\partial}{\partial \phi} u_k(x) \right)^2}_{=\lambda} + 2 \sum_{k=0}^{2\ell} \frac{\partial}{\partial \theta} u_k(x) \cdot \frac{1}{\sin\theta}\frac{\partial}{\partial \phi} u_k(x) \end{aligned}$$

Therefore

$$\langle \vec{v}(x), \vec{w}(x) \rangle = \sum_{k=0}^{2\ell} \frac{\partial}{\partial \theta} u_k(x) \cdot \frac{1}{\sin\theta}\frac{\partial}{\partial \phi} u_k(x) = 0$$

as proposed.

Thus proposition 1 has been shown.□

A little bit analysis

The square of distance function $d^2(u)$ is a special case of a family of functions

$$g_\alpha(u) := \lambda u^2(x) + \alpha |\nabla u(x)|^2$$

for which we want to show some properties.

Lemma 4:

(1) If $y \in S^2$ is a local maximum of $\lambda u(x)^2 + \alpha |\nabla u(x)|^2$, then $|\nabla u(y)|^2 = 0$, for all $0 \leq \alpha < 1$.

(2) If $y \in S^2$ is a minimum of $\lambda u^2(x) + \alpha |\nabla u(x)|^2$, then $|\nabla u(y)|^2 = 0$, for all $\alpha > \frac{\lambda}{\lambda - 1}$.

Proof:

We use a method which can be found in [SR].

(1) For $\alpha = 0$ the proposition is trivial. So let $0 < \alpha < 1$.
Let $y \in S^2$ be a maximum of $g_\alpha(x)$. Thus $|\nabla g_\alpha(y)|^2 = 0$. We have to show, that $|\nabla u(y)|^2 = 0$.
So suppose that $|\nabla u(y)|^2 \neq 0$. Then we can choose the coordinate system, so that

$$u_\theta(y) \neq 0 \text{ and } u_\phi(y) = 0$$

where u_θ and u_ϕ denote $\frac{\partial}{\partial\theta}u$ and $\frac{\partial}{\partial\phi}u$ respectively.
So we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial\theta} g_\alpha(y) \\ &= 2\lambda u(y)u_\theta(y) + 2\alpha u_\theta(y)u_{\theta\theta}(y) + 2\alpha\frac{1}{\sin\theta}\underbrace{u_\phi(y)}_{=0}\frac{\partial}{\partial\theta}\big(\frac{1}{\sin\theta}u_\phi(y)\big) \end{aligned}$$

Hence

$$\lambda u(y)u_\theta(y) + \alpha u_\theta(y)u_{\theta\theta}(y) = 0$$

and

$$u_{\theta\theta}(y) = -\frac{\lambda}{\alpha}u(y) \tag{1}$$

since $u_\theta(y) \neq 0$ by assumption. Analogously

$$\begin{aligned} 0 &= \frac{\partial}{\partial\phi} g_\alpha(y) \\ &= 2\lambda u(y)u_\phi(y) + 2\alpha\frac{1}{\sin\theta}u_\theta(y)u_{\theta\phi}(y) + 2\alpha\frac{1}{\sin^3\theta}u_\phi(y)u_{\phi\phi}(y) \end{aligned}$$

and hence

$$u_\theta(y)u_{\theta\phi}(y) = 0 \text{ and } u_{\theta\phi}(y) = 0 \tag{2}$$

Now a tedious but simple calculation (one just has to calculate both sides) yields by using (1) and (2)

$$\begin{aligned} \frac{1}{2}\Delta g_\alpha(y) =& \lambda|\nabla u|^2 + \underbrace{\lambda u\Delta u}_{=-\lambda^2u^2} + \alpha\underbrace{\nabla u\cdot\nabla\Delta u}_{=-\lambda|\nabla u|^2} + \alpha\underbrace{u_{\theta\theta}^2}_{\frac{\lambda^2}{\alpha^2}u^2} \\ &+ \frac{2\alpha}{\sin^2\theta}\underbrace{u_{\theta\phi}^2}_{=0} + \alpha\frac{\cos^2\theta+1}{\sin^4\theta}\underbrace{u_\phi^2}_{=0} - 4\alpha\frac{\cos\theta}{\sin^3\theta}\underbrace{u_\phi}_{=0}u_{\theta\phi} \\ &+ \frac{\alpha}{\sin^4\theta}u_{\phi\phi}^2 + 2\alpha\frac{\cos\theta}{\sin^3\theta}u_\theta u_{\phi\phi} + \frac{\alpha}{\sin^2\theta}u_\theta^2 \end{aligned}$$

Since

$$\begin{aligned}
&\frac{\alpha}{\sin^4\theta}u_{\phi\phi}^2 + 2\alpha\frac{\cos\theta}{\sin^3\theta}u_\theta u_{\phi\phi} + \frac{\alpha}{\sin^2\theta}u_\theta^2 \\
&\qquad = \alpha\Big(\frac{1}{\sin^2\theta}u_{\phi\phi} + \frac{\cos\theta}{\sin\theta}u_\theta\Big)^2 - \alpha\frac{\cos^2\theta}{\sin^2\theta}u_\theta^2 + \frac{\alpha}{\sin\theta}u_\theta^2 \\
&\qquad = \alpha\big(\Delta u - u_{\theta\theta}\big)^2 + \alpha\left(\frac{1-\cos^2\theta}{\sin^2\theta}\right)u_\theta^2 \\
&\qquad = \alpha(-\lambda u + \frac{\lambda}{\alpha}u)^2 + \alpha u_\theta^2
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{2}\Delta g_\alpha(y) = \\
&= \underbrace{(1-\alpha)}_{>0}\lambda\underbrace{|\nabla u|^2}_{\neq 0} + \underbrace{(\frac{1}{\alpha}-1)}_{>0}\lambda^2u^2 + \underbrace{\alpha(\frac{\lambda}{\alpha}u - \lambda u)^2}_{\geq 0} + \underbrace{\alpha u_\theta^2}_{>0}
\end{aligned}$$

Therefore

$$\Delta g_\alpha(y) > 0$$

and hence by the maximum principle y cannot be a maximum of g_α, a contradiction to our assumption.

(2) Let $y \in S^2$ be a minimum of $g_\alpha(x) := \lambda u^2(x) + \alpha|\nabla u(x)|^2$. Analogously to (1) suppose $|\nabla u(y)|^2 \neq 0$.
Again we can choose the coordinate system, so that

$$u_\theta(y) \neq 0 \text{ and } u_\phi(y) = 0$$

and obtain

$$\begin{aligned}
\tfrac{1}{2}\Delta g_\alpha(y) &= (1-\alpha)\lambda\underbrace{|\nabla u|^2}_{=u_\theta^2} + (\tfrac{1}{\alpha}-1)\lambda^2u^2 + \alpha(\tfrac{\lambda}{\alpha}u - \lambda u)^2 + \alpha u_\theta^2 \\
&= \underbrace{\big((1-\alpha)\lambda + \alpha\big)}_{<0 \text{ for } \alpha > \frac{\lambda}{\lambda-1}}\underbrace{u_\theta^2}_{\neq 0} + \underbrace{\alpha(\tfrac{1}{\alpha}-1)^3}_{<0 \text{ for } \alpha>1}\lambda^2u^2
\end{aligned}$$

But now

$$\Delta g_\alpha(y) < 0 \text{ for } \alpha > \tfrac{\lambda}{\lambda-1}$$

and hence by the maximum principle y cannot be a minimum of g_α, a contradiction to our assumption.□

As a corollary we obtain a result, which already can be found in [SR].

Lemma 5:
Denote $u_M = \max_{x \in S^2} |u(x)|$. Then

$$\lambda u(x)^2 + \alpha |\nabla u(x)|^2 \leq \lambda u_M^2 \qquad \text{for all } 0 \leq \alpha \leq 1$$

Proof:
Let $0 \leq \alpha < 1$. Then by definition

$$\lambda u_M^2 = \lambda \max_{x \in S^2} |u(x)|^2 \leq \max_{x \in S^2} \{\lambda u^2(x) + \alpha |\nabla u(x)|^2\} = \lambda u^2(y) + \alpha |\nabla u(y)|^2$$

for a point $y \in S^2$ since S^2 is compact. Because of lemma 4(1), $|\nabla u(y)|^2$ must be zero. Therefore we further obtain

$$\lambda u^2(y) + \alpha |\nabla u(y)|^2 = \lambda u^2(y) \leq \lambda u_M^2$$

and so

$$\lambda u^2(x) + \alpha |\nabla u(x)|^2 \leq \max_{x' \in S^2} \{\lambda u^2(x') + \alpha |\nabla u(x')|^2\} \leq \lambda u_M^2$$

for all $0 \leq \alpha < 1$. Because of the continuity of $\alpha \mapsto g_\alpha(x)$ the lemma also is true for $\alpha = 1$.□

From proposition 1 we can derive another similar inequality.

Lemma 6:

$$\lambda u^2(x) + \alpha |\nabla u(x)|^2 \leq \lambda \eta^2 \|u\|_2^2 \qquad \text{for all } 0 \leq \alpha \leq 2$$

Proof:
It is sufficient to show the lemma for $\alpha = 2$. Let $\vec{c}_u$ be the coefficient vector or u. Then because of proposition 1 and Bessel's inequality

$$\|u\|_2^2 = |\vec{c}_u|^2 \geq \langle \vec{c}_u, \vec{u}(x) \rangle^2 + \langle \vec{c}_u, \vec{v}(x) \rangle^2 + \langle \vec{c}_u, \vec{w}(x) \rangle^2$$

Since by the proof of proposition 2 the right hand side of this inequality equals to

$$\frac{1}{\eta^2} \left(u^2(x) + \frac{2}{\lambda} |\nabla u(x)|^2 \right)$$

the result follows.□

But now we return to the distance function.

Proof of theorem 1

From lemma 4(2) we can conclude that

$$\alpha \mapsto \min_{x\in S^2} g_\alpha(u,x)$$

is constant for $\alpha > \frac{\lambda}{\lambda-1}$. Hence by continuity

$$d^2(u) = \frac{4\pi}{2\ell+1}\frac{1}{\lambda}\min_{x\in S^2} g_2(u,x) = \frac{4\pi}{2\ell+1}\frac{1}{\lambda}\min_{x\in S^2} g_{\frac{\lambda}{\lambda-1}}(u,x)$$

Thus we have

$$\begin{aligned}\int_{S^2} d^2(u,x)d\sigma &= \tfrac{4\pi}{2\ell+1}\Big[\underbrace{\int_{S^2} u^2(x)d\sigma}_{=\|u\|^2} + \tfrac{\lambda}{\lambda-1}\tfrac{1}{\lambda}\underbrace{\int_{S^2} |\nabla u(x)|^2 d\sigma}_{=-\int_{S^2} u\Delta u d\sigma=\lambda\|u\|^2}\Big] \\ &= \tfrac{4\pi}{2\ell+1}(2+\tfrac{1}{\lambda-1})\|u\|^2\end{aligned}$$

Therefore

$$4\pi d^2(u) = \min_{x\in S^2} d^2(u,x)\int_{S^2} d\sigma \le \int_{S^2} d^2(u,x)d\sigma = \tfrac{4\pi}{2\ell+1}(2+\tfrac{1}{\lambda-1})\|u\|^2$$

and

$$d(u) \le \sqrt{\frac{1}{2\ell+1}(2+\frac{1}{\lambda-1})}\,\|u\|$$

as claimed.□

Proof of theorem 2

Without loss we can assume u to be a zonal harmonic with the pole of the coordinate system as its pole. Then $\frac{\partial u}{\partial \phi} = 0$, i.e. u does not depend on the coordinate ϕ and can be written as (a multiple of)

$$u(\theta,\phi) = P_\ell(\cos\theta) \quad \text{for some } \ell \in \mathbb{N}$$

Our goal is to calculate $d^2(u)$. By lemma 4(2) we have to estimate the least value of u^2 with $|\nabla u| = 0$.

For this purpose we use the function $g_1(x) = \lambda u(x)^2 + |\nabla u(x)|^2$, for which we can show

Lemma 7:

$$\frac{\partial g_1}{\partial \theta}(x) = -2\cot\theta\, u_\theta(x)^2$$

where $u_\theta = \frac{\partial u}{\partial \theta}$.

Proof:
Since $\frac{\partial u}{\partial \phi} \equiv 0$ and we have $g_1 = \lambda u^2 + |\nabla u|^2 = \lambda u^2 + u_\theta^2$ and $\lambda u = -\Delta u = -u_{\theta\theta} - \cot\theta\, u_\theta$. Thus

$$\frac{\partial g_1}{\partial \theta} = 2\lambda u u_\theta + 2u_\theta u_{\theta\theta} = 2u_\theta(\lambda u + u_{\theta\theta}) = -2\cot\theta\, u_\theta^2$$

as proposed.□

It follows immediately that the maxima of $|P_\ell(\cos\theta)|$ are increasing for increasing $|\cos\theta|$ (see also [SZ], theorem 7.3.1).

Since $|P_\ell|$ has a maximum at 0 for even ℓ we can calculate $d^2(u)$ immediately in this case:

Each normalized zonal harmonic u can be written as

$$u(x) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

Now we already know, that u^2 has its least maximum at the equator. Hence by proposition 2

$$d^2(u) = |P_\ell(0)|^2$$

From the explicit expression of the Legendre-polynomials P_ℓ we get

$$|P_\ell(0)| = \frac{1}{2^{2k}}\binom{2k}{k} \qquad \text{for } \ell = 2k$$

By means of Stirling's formula

$$n! = \sqrt{2\pi}\cdot\sqrt{n}\cdot n^n \cdot \exp\left(-n + \frac{\delta}{12n}\right) \qquad \text{for a } 0 < \delta < 1$$

we obtain

$$
\begin{aligned}
&= \frac{1}{2^{2k}} \binom{2k}{k} \\
&= \frac{1}{2^{2k}} \frac{(2k)!}{(k!)^2} \\
&= \frac{\sqrt{2\pi} \cdot \sqrt{2k} \cdot (2k)^{2k} \cdot \exp\left(-2k + \frac{\delta_1}{24k}\right)}{2^{2k} \cdot \left(\sqrt{2\pi} \cdot \sqrt{k} \cdot k^k \cdot \exp\left(-k + \frac{\delta_2}{12k}\right)\right)^2} \qquad \text{for proper } 0 < \delta_1, \delta_2 < 1 \\
&= \frac{\sqrt{2\pi} \cdot \sqrt{2k} \cdot 2^{2k} \cdot k^{2k} \cdot \exp\left(-2k + \frac{\delta_1}{24k}\right)}{2^{2k} \cdot 2\pi \cdot k \cdot k^{2k} \cdot \exp\left(-2k + \frac{\delta_2}{6k}\right)} \\
&= \sqrt{\frac{1}{2\pi}} \cdot \frac{\sqrt{2k}}{k} \cdot \exp\left(\frac{\delta_1}{24k} - \frac{\delta_2}{6k}\right) \\
&= \sqrt{\frac{2}{\pi\ell}} \cdot \exp\left(\frac{\delta_1}{12\ell} - \frac{\delta_2}{3\ell}\right)
\end{aligned}
$$

$$
d^2(u) = |P_\ell(0)|^2 = \frac{2}{\pi\ell} \cdot \exp\left(\frac{2\delta_1}{12\ell} - \frac{2\delta_2}{3\ell}\right) \qquad \text{for proper } 0 < \delta_1, \delta_2 < 1
$$

From this the first part of theorem 2 follows.

The case of odd ℓ is a little bit more tricky.

From the definitions of $d^2(u, x)$ and $g_\alpha(u, x)$ we have

$$
d^2(u, x) = \frac{4\pi}{2\ell + 1} \frac{1}{\lambda} g_2(u, x) \geq \frac{4\pi}{2\ell + 1} \frac{1}{\lambda} g_1(u, x)
$$

and hence

$$
d^2(u) = \min_{x \in S^2} d^2(u, x) \geq \frac{4\pi}{2\ell + 1} \frac{1}{\lambda} \min_{x \in S^2} g_1(u, x)
$$

From lemma 7 we know that $g_1(u)$ takes its minimum where $\cot\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2}$, i.e. at the equator to the pole of the zonal harmonic. But there $u = 0$. Therefore

$$
\min_{x \in S^2} g_1(u, x) = \frac{2\ell + 1}{4\pi} \left(\frac{\partial}{\partial\theta} P_l(\cos\theta)|_{\theta = \frac{\pi}{2}}\right)^2 = \frac{2\ell + 1}{4\pi} |P_l^1(0)|^2
$$

and hence

$$
d^2(u) \geq \frac{4\pi}{2\ell + 1} \frac{1}{\lambda} \frac{2\ell + 1}{4\pi} |P_l^1(0)|^2 = \frac{1}{\lambda} |P_l^1(0)|^2
$$

On the other hand we know from lemma 4(2) that $g_\alpha(u, x)$ cannot have a minimum at $\cos\theta = 0$ for $\alpha > \frac{\lambda}{\lambda - 1}$ since $\nabla u|_{\theta = 0} \neq 0$. Hence by continuity

$$
\min_{x \in S^2} g_2(u, x) = \min_{x \in S^2} g_{\frac{\lambda}{\lambda - 1}}(u, x) \leq g_{\frac{\lambda}{\lambda - 1}}(u, \theta = \tfrac{\pi}{2})
$$

And so

$$d^2(u) \le \frac{4\pi}{2\ell+1}\frac{1}{\lambda}\frac{2\ell+1}{4\pi}\frac{\lambda}{\lambda-1}|P_l^1(0)|^2 = \frac{1}{\lambda-1}|P_l^1(0)|^2$$

Therefore

$$\frac{1}{\lambda}|P_l^1(0)|^2 \le d^2(u) \le \frac{1}{\lambda-1}|P_l^1(0)|^2$$

From the explicit expression of the Legendre-polynomial P_ℓ we have

$$\begin{aligned}
|P_\ell^1(0)| &= \frac{1}{2^\ell}\binom{\ell}{\frac{\ell-1}{2}}\underbrace{\binom{\ell+1}{\ell}}_{=\ell+1} \\
&= \frac{\ell+1}{2^\ell}\frac{\ell!}{(\frac{\ell-1}{2})!(\frac{\ell+1}{2})!} \\
&= \frac{\ell+1}{2^{\ell+1}}\frac{\ell!}{(\frac{\ell+1}{2})!^2} \\
&= (\ell+1)|P_{\ell+1}(0)|
\end{aligned}$$

Since $\ell+1$ is even we can use the results of the first part of the theorem.

$$d^2(u) \ge \frac{1}{\lambda}(\ell+1)^2|P_{\ell+1}(0)|^2 > \frac{2}{\ell\pi}\exp(-\frac{2}{3(\ell+1)})$$

and

$$d^2(u) \le \frac{1}{\lambda-1}(\ell+1)^2|P_{\ell+1}(0)|^2 < (\frac{2}{\ell\pi} + \frac{2}{\pi}\frac{1}{\ell^3+\ell^2-\ell})\exp(\frac{1}{6(\ell+1)})$$

as proposed.□

Some further results

The first result can be found in [SR]. It can be used to derive a lower bound on the size of nodal domains.

Theorem 3:
By $\rho(x_1, x_2)$ we denote the geodesic distance between two points $x_1, x_2 \in S^2$ on the sphere. Then

(1) $$\left|\arcsin\left(\frac{u(x_1)}{u_M}\right) - \arcsin\left(\frac{u(x_2)}{u_M}\right)\right| \le \sqrt{\lambda}\rho(x_1, x_2)$$

(2) $$\left|\arcsin\left(\frac{u(x_1)}{\eta\|u\|}\right) - \arcsin\left(\frac{u(x_2)}{\eta\|u\|}\right)\right| \le \sqrt{\frac{\lambda}{2}}\rho(x_1, x_2)$$

Proof: [SR]

(1) Let $q(t)$ be a constant speed curve from x_1 to x_2 along a great circle, with $q(0) = x_1$, $q(1) = x_2$ and length $\leq \pi$. Then the parameter t depends linearly on the arc length and

$$|\dot{q}(t)| = \rho(x_1, x_2) =: \rho$$

Set $M(t) = u\big(q(t)\big)$. Then

$$M'(t) = \nabla u\big(q(t)\big) \cdot \dot{q}(t) \leq \big|\nabla u\big(q(t)\big)\big| \cdot \big|\dot{q}(t)\big|$$

From lemma 5 we have

$$\begin{aligned}
|\nabla u|^2 &\leq \lambda(u_M^2 - u^2) \\
\frac{\big|\nabla u\big(q(t)\big)\big|}{\sqrt{u_M^2 - u^2(q(t))}} &\leq \sqrt{\lambda} \\
\frac{\big|\nabla u\big(q(t)\big)\big| \cdot \big|\dot{q}(t)\big|}{u_M\sqrt{1 - \frac{u^2(q(t))}{u_M^2}}} &\leq \sqrt{\lambda}\big|\dot{q}(t)\big| = \sqrt{\lambda}\,\rho \\
\frac{1}{u_M}\frac{M'(t)}{\sqrt{1 - \frac{M^2(t)}{u_M^2}}} &\leq \sqrt{\lambda}\,\rho
\end{aligned}$$

By integrating along q we obtain By integrating along q we obtain

$$\int_0^1 \frac{1}{u_M}\frac{M'(t)dt}{\sqrt{1 - \frac{M^2(t)}{u_M^2}}} \leq \int_0^1 \sqrt{\lambda}\,\rho\,dt = \sqrt{\lambda}\,\rho$$

We calculate the left hand side by substitution which yields

$$\begin{aligned}
&= \int_{M(0)}^{M(1)} \frac{1}{u_M}\frac{dM}{\sqrt{1 - \frac{M^2}{u_M^2}}} \\
&= \int_{\frac{u(x_1)}{u_M}}^{\frac{u(x_2)}{u_M}} \frac{dM}{\sqrt{1 - M^2}} \\
&= \arcsin\left(\frac{u(x_2)}{u_M}\right) - \arcsin\left(\frac{u(x_1)}{u_M}\right)
\end{aligned}$$

Hence

$$\arcsin\left(\frac{u(x_2)}{u_M}\right) - \arcsin\left(\frac{u(x_1)}{u_M}\right) \leq \sqrt{\lambda}\,\rho$$

By changing the role of x_1 and x_2 we get the proposition immediately.

(2) The second inequality can be derived in the same way from lemma 6.□

Next we use lemma 5 to show a simple lower bound for the maxima of spherical harmonics.

Theorem 4:

$$u_M \geq \sqrt{\frac{1}{2\pi}} \|u\|$$

Proof:
By lemma 5 it follows that

$$\begin{aligned} u_M^2 &= \tfrac{1}{4\pi} \int_{S^2} u_M^2(x) d\sigma \\ &\geq \tfrac{1}{4\pi} \int_{S^2} \left(u^2(x) + \tfrac{1}{\lambda} |\nabla u(x)|^2\right) d\sigma \\ &= \tfrac{1}{4\pi} \Big(\underbrace{\int_{S^2} u^2(x) d\sigma}_{=\|u\|^2} + \underbrace{\tfrac{1}{\lambda} \int_{S^2} |\nabla u(x)|^2 d\sigma}_{=\lambda\|u\|^2} \Big) \\ &= \tfrac{1}{2\pi} \|u\|^2 \end{aligned}$$

At last we want to derive an estimate for the value of saddle points of a spherical harmonic.

Theorem 5:
For $\ell \geq 2$ let $y \in S^2$ be a saddle point of u. Then

$$u^2(y) < \frac{1}{3} \eta^2 \|u\|^2$$

Remark:
For an arbitrary stationary point y of a spherical harmonic we have $u^2(y) \leq \eta^2 \|u\|^2$ and this estimate is sharp (see the first remark).

Proof of theorem 5:
For proving this theorem, we use the base functions Y_ℓ^m. Each spherical harmonic can be written as

$$u(\theta, \varphi) = \sum_{i=-\ell}^{\ell} \alpha_i Y_\ell^i(\theta, \varphi)$$

Since y is a saddle point of u, the principal curvatures at y — i.e. the eigenvalues of the Hessian of u, which are equal to the greatest and lowest curvature at y respectively — cannot have the same sign.

Without loss we now can assume y as pole of the S^2. To calculate the curvature at y in any direction, we have to estimate

$$\left(\frac{\partial}{\partial\theta}\right)^2 Y_\ell^m(\theta,\varphi)\Big|_{\theta=0} = \left(\left(\frac{d}{d\theta}\right)^2 P_\ell^{|m|}(\cos\theta)\right) c_{\ell m} \begin{cases}\cos m\varphi \\ \sin m\varphi\end{cases}$$

Now

$$P_\ell^m(\cos\theta) = (\sin\theta)^m \left(\frac{d}{dx}\right)^m P_\ell(\cos\theta)$$

then we obtain

$$\begin{aligned}
\left(\frac{d}{d\theta}\right)^2 P_\ell^0(\cos\theta)\Big|_{\theta=0} &= -\underbrace{\cos\theta}_{=1} P_\ell'(\cos\theta)\Big|_{\theta=0} + \underbrace{\sin^2\theta}_{=0} P_\ell''(\cos\theta)\Big|_{\theta=0} \\
&= -\,P_\ell'(1) \\
\left(\frac{d}{d\theta}\right)^2 P_\ell^1(\cos\theta)\Big|_{\theta=0} &= \left(\frac{d}{d\theta}\right)^2 \sin\theta P_\ell'(\cos\theta)\Big|_{\theta=0} \\
&= \big(-\sin\theta P_\ell'(\cos\theta) - \cos\theta\sin\theta P_\ell''(\cos\theta) \\
&\quad - 2\sin\theta\cos\theta P_\ell''(\cos\theta) + \sin^3\theta P_\ell'''(\cos\theta)\big)\Big|_{\theta=0} \\
&= 0 \\
\left(\frac{d}{d\theta}\right)^2 P_\ell^2(\cos\theta)\Big|_{\theta=0} &= \left(\frac{d}{d\theta}\right)^2 \sin^2\theta P_\ell''(\cos\theta)\Big|_{\theta=0} \\
&= \big(2\cos^2\theta P_\ell''(\cos\theta) + 2\cos\theta\sin^2\theta P_\ell'''(\cos\theta) \\
&\quad + \sin^4\theta P_\ell''''(\cos\theta)\big)\Big|_{\theta=0} \\
&= 2P_\ell''(1)
\end{aligned}$$

Since each term of $\left(\frac{d}{d\theta}\right)^2 P_\ell^m(\cos\theta)$ contains $(\sin\theta)$ for $m > 2$

$$\left(\frac{d}{d\theta}\right)^2 P_\ell^m(\cos\theta)\Big|_{\theta=0} = 0 \qquad \text{for } m > 2$$

Therefore we obtain for the principal curvature of u at y

$$\alpha_0\, c_{\ell 0}\left(-P_\ell'(1)\right) \;\pm\; \sqrt{\alpha_{-2}^2 + \alpha_2^2}\, c_{\ell 2}\left(2P_\ell''(1)\right)$$

These have different sign if and only if

$$\alpha_0^2\, c_{\ell 0}^2\big(P_\ell'(1)\big)^2 \le (\alpha_{-2}^2 + \alpha_2^2) c_{\ell 2}^2\big(2P_\ell''(1)\big)^2$$

and hence if and only if

$$\frac{\alpha_0^2}{\alpha_{-2}^2+\alpha_2^2}\frac{c_{\ell 0}^2}{4c_{\ell 2}^2}\left(\frac{P_\ell'(1)}{P_\ell''(1)}\right)^2 \le 1$$

Now

$$\frac{c_{\ell 0}^2}{4c_{\ell 2}^2} = \frac{\frac{2\ell+1}{4\pi}}{4\frac{2\ell+1}{4\pi}2\frac{(\ell-2)!}{(\ell+2)!}} = \frac{(\ell+2)!}{8(\ell-2)!} = \frac{1}{8}(\ell+2)_4 = \frac{1}{8}(\ell+2)(\ell+1)\ell(\ell-1)$$

To calculate the ratio between $P_\ell'(1)$ and $P_\ell''(1)$ we notice that P_ℓ satisfies the ODE

$$(1-x^2)f'' - 2xf' + \lambda f = 0$$

with $\lambda = \ell(\ell+1)$. By differentiation and setting $x=1$ we obtain

$$-2f''(1) - 2f'(1) - 2f''(1) + \lambda f'(1) = 0$$

$$-4f''(1) + (\lambda-2)f'(1) = 0$$

Thus

$$\left(\frac{P_\ell'(1)}{P_\ell''(1)}\right)^2 = \left(\frac{f'(1)}{f''(1)}\right)^2 = \frac{16}{(\lambda-2)^2}$$

since $P_\ell'(1) \neq 0$ (can easily be seen from the ODE) — and hence $P_\ell''(1) \neq 0$ — and $\lambda = \ell(\ell+1) \ge 6$ by assumption.

Therefore

$$\frac{\alpha_0^2}{\alpha_{-2}^2+\alpha_2^2}\cdot\frac{1}{8}(\ell+2)_4\cdot\frac{16}{(\lambda-2)^2} \le 1$$

if $y =$ pole is a saddle point of u, which is equivalent to

$$\alpha_0^2 \le \frac{1}{2}\underbrace{\frac{\big(\ell(\ell+1)-2\big)^2}{(\ell+2)_4}}_{<1}(\alpha_{-2}^2+\alpha_2^2) < \frac{1}{2}(\alpha_{-2}^2+\alpha_2^2) \tag{3}$$

since

$$\big(\ell(\ell+1)-2\big)^2 = \ell^4+2\ell^3-3\ell^2-4\ell-4 < \ell^4+2\ell^3-\ell^2-2\ell = (\ell+2)_4 \quad \text{if } \ell \ge 2$$

Now

$$\alpha_0^2 + (\alpha_{-2}^2+\alpha_2^2) \le \sum_{k=-\ell}^{\ell}\alpha_k^2 = |\vec{c}|^2 = \|u\|^2$$

Together with (3) we obtain

$$\alpha_0^2 < \frac{1}{3}\|u\|^2$$

Since $u_m(pole) = 0$ for $m \neq 0$ ($\sin\theta = 0$ at $\theta = 0$!) and $u_0(pole) = \eta$ (see remark 1 about the eigenspace)

$$u^2(pole) = \alpha_0^2\cdot u_0^2(pole) = c_0^2\cdot\eta^2 < \frac{1}{3}\eta^2\|u\|^2$$

as claimed.□

Appendix A

We prove the following propositions from the chapter about conics

Proposition 1:

Let p and q be two irreducible conics which have four real common points. Suppose p and q are orthogonal to each other, i.e. their tangents at these points are orthogonal. Then (without loss) the center of p is the saddle of q and the saddle of p must be inside of q. Furthermore neither p nor q is a circle.
If moreover the stationary points of p and q coincide, then the center of one is the saddle of the other and vice versa.

Corollary 1:

If two conics have a common center, they cannot intersect orthogonally.

Corollary 2:

If three conics p, q_1 and q_2 intersect each other orthogonally, where p intersects each of the other conics q_i in four points, then q_1 and q_2 have at most two (pairs of) common points.

Proposition 2:

If three conics p, q_1 and q_2 intersect each other orthogonally, where p intersects each of the other conics q_i in four points, then q_1 and q_2 have no common points.

Proof of proposition 1

By a proper choice of the coordinate system (i.e. take the eigenvectors of A_P as coordinate axes), P can be written as

$$P = \alpha x^2 + \beta y^2 + \gamma z^2$$

where

$$\alpha \geq \beta > 0, \ \gamma < 0 \tag{1}$$

are the eigenvalues of A_P with eigenvectors $e_1 = (0, 0, 1)$, $e_2 = (0, 1, 0)$, $e_3 = (1, 0, 0)$. Under this assumptions p cannot have any (real) point on the line $z = 0$ (i.e. the line with equation $z = 0$).

Q always can be written as

$$Q = ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

with real coefficients, and by assumption it is irreducible. In the following λ_1, λ_2 and λ_3 always denote the eigenvalues of A_Q.

Now if p, q have four common points and if their tangents are orthogonal at these, then the polynomial of degree two

$$R = \tfrac{1}{2}\nabla P \cdot \nabla Q = 2\alpha ax^2 + 2\beta by^2 + 2\gamma cz^2 + (\alpha+\beta)dxy + (\alpha+\gamma)exz + (\beta+\gamma)fyz$$

must be zero at these points also. Since p and q are different, they determine a pencil of conics, which contains r ([BE2], Theorem 16.4.8, p. 185), i.e.

$$R = \mu P + \nu Q$$

for a proper choice of $\mu, \nu \in \mathbb{R}$. By comparison of coefficients we yield

$$\begin{aligned} \mu\alpha + \nu a &= 2\alpha a \\ \mu\beta + \nu b &= 2\beta b \\ \mu\gamma + \nu c &= 2\gamma c \end{aligned} \tag{2}$$

$$\begin{aligned} \nu d &= (\alpha+\beta)d \\ \nu e &= (\alpha+\gamma)e \\ \nu f &= (\beta+\gamma)f \end{aligned} \tag{3}$$

First we show, that there is at least one point which is a stationary point of both p and q, i.e. one of the eigenvectors of e_1, e_2 or e_3 von A_P must be eigenvector of A_Q. Since

$$A_Q = \begin{pmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{pmatrix}$$

it remains to show

Lemma 1:
At least two of the parameters d, e, f must be zero.

Proof:

Indirect: Suppose at most one of the parameters is zero.

Case: $d \neq 0, e \neq 0$ or $d \neq 0, f \neq 0$

In the first case we obtain from (3) that $\nu = \alpha + \beta$ and $\nu = \alpha + \gamma$ hence $\beta = \gamma$, a contradiction to (1).

Analogously we obtain in the second case $\alpha = \gamma$, again a contradiction to (1).

Case: $d = 0,\ e \neq 0, f \neq 0$

Again we obtain from (3) $\alpha = \beta$ and hence from (2)

$$0 = a(\nu - 2\alpha) + \alpha\mu$$
$$0 = b(\nu - 2\beta) + \beta\mu = b(\nu - 2\alpha) + \alpha\mu$$

and

$$a(\nu - 2\alpha) = b(\nu - 2\alpha)$$

Now either $\nu = 2\alpha = \alpha + \gamma$ (from (3)) and hence $\gamma = \alpha$, a contradiction, or $a = b$.

Thus both p and q must be circles, which intersect in four points orthogonally. But since a circle is uniquely determined by three points, this is a contradiction.□

Next we show that this common stationary point is the center of one conic and the saddle point of the other.

First we deal with the case where all stationary points of p and q coincide, i.e.

Case: $d = e = f = 0$

Then Q reduces to

$$Q = ax^2 + by^2 + cz^2$$

Lemma 2:

q is uniquely determined by p and a common point $(s, t, 1) \in p$.

Proof:

Suppose there would be another conic $q' \neq p$ with the same properties as q. The four points $(\pm s, \pm t, 1)$ are then common to p, q and q' and hence p must be contained in the pencil determined by q and q'. Since q and q' are tangent to p so every conic is contained in this pencil, especially this must be true for p itself, a contradiction.□

Therefore Q can be uniquely written as

$$Q = \begin{vmatrix} x^2 & y^2 & z^2 \\ s^2 & t^2 & 1 \\ \alpha s^2 & \beta t^2 & \gamma \end{vmatrix}$$

From this we can calculate the coefficients of Q, which (in this case) coincide with the eigenvalues of A_Q. From (1) we get their signs:

$$\begin{aligned} a &= t^2(\beta - \gamma) \; > 0 \\ b &= s^2(\gamma - \alpha) \; < 0 \\ c &= s^2 t^2(\alpha - \beta) \; > 0 \end{aligned}$$

As can easily be seen, $\alpha \neq \beta$. Otherwise $c = 0$ and hence q would be singular and decomposable. Therefore p owns a saddle point, namely $e_2 = (0, 1, 0)$ and it must be the center of q. By changing the role of p and q we obtain in this case, that the center of p is the saddle point of q, too. This finishes the proof of lemma 2.

Now all the cases remain, where p and q have only one common stationary point.

Case: $d \neq 0$ und $e = f = 0$
Then $e_1 = (0, 0, 1)$ — the center of p — is eigenvector of A_Q.

We show: e_1 is the saddle point of q.

From (2) and (3) we obtain

$$\begin{aligned} \nu &= \alpha + \beta \\ \mu\alpha &= (\alpha - \beta)a \\ \mu\beta &= (\beta - \alpha)b \\ \mu\gamma &= (2\gamma - \alpha - \beta)c \end{aligned}$$

Since $(2\gamma - \alpha - \beta) < 0$, $\mu \neq 0$. Otherwise $c = 0$ and since c is the eigenvalue to the eigenvector e_1 (see below), Q would be decomposable. Moreover $\alpha \neq \beta$, else $\mu = 0$. We yield from (1)

$$\begin{aligned} a &= \frac{\alpha}{\alpha - \beta} \; > 0 \\ b &= \frac{\beta}{\beta - \alpha} \; < 0 \\ c &= \frac{\gamma}{2\gamma - \alpha - \beta} \; > 0 \end{aligned} \tag{4}$$

We obtain for the characteristic polynomial of A_Q

$$\left((a-\lambda)(b-\lambda)-\tfrac{1}{4}d^2\right)(c-\lambda)=0$$

and hence for the eigenvalues

$$\begin{aligned}\lambda_2 &= c\\ \lambda_{1,3} &= \tfrac{a+b}{2} \mp \sqrt{(\tfrac{a+b}{2})^2 - ab + \tfrac{1}{4}d^2}\\ &= \tfrac{1}{2}\left((a+b) \mp \sqrt{(a-b)^2+d^2}\right)\\ &= \tfrac{1}{2}\left(1 \mp \sqrt{\left(\tfrac{\alpha+\beta}{\alpha-\beta}\right)^2 + d^2}\right)\end{aligned}$$

Since $(a-b)^2 > (a+b)^2$, $\lambda_1 < 0$. Since $\alpha+\beta > 0$, $c = \frac{\gamma}{2\gamma-\alpha-\beta} < \frac{1}{2}$, and hence $\lambda_2 = c < \lambda_3$.

Therefore, the eigenvalues of A_Q are different, i.e. the saddle point of q exists, and (since λ_2 is the eigenvalue to the eigenvector e_1) this saddle point is e_1, the center of p as proposed. Moreover together with (4) we find, that neither p nor q is a circle.

It remains to show, that $e_2 = (0,1,0)$ — the saddle of p — is inside of q. Since there is only one negative eigenvalue of A_Q, and since we have $Q(e_2) = b < 0$, this follows immediately.

Case: $e \neq 0$, $d = f = 0$

Then $e_2 = (0,1,0)$ — the saddle point[1] of p — is eigenvector of A_Q.

We show: e_2 is the center of q.

Again we obtain from (1), (2) and (3)

$$\begin{aligned}\nu &= \alpha+\gamma\\ \mu\alpha &= (\alpha-\gamma)a\\ \mu\beta &= (2\beta-\alpha-\gamma)b\\ \mu\gamma &= (\gamma-\alpha)c\end{aligned}$$

Since again $\mu \neq 0$, we have

$$2\beta-\alpha-\gamma \neq 0$$

[1] That this really is a saddle will be shown at the end of the proof of this case.

and

$$\begin{aligned} a &= \frac{\alpha}{\alpha - \gamma} \; > 0 \\ b &= \frac{\beta}{2\beta - \alpha - \gamma} \\ c &= \frac{\gamma}{\gamma - \alpha} \; > 0 \end{aligned} \tag{5}$$

We show:

$$b < 0 \tag{6}$$

$$e^2 < -4\frac{\alpha\gamma}{(\alpha-\gamma)^2} \tag{7}$$

From the characteristic polynomial of A_Q we yield

$$\begin{aligned} \lambda_1 &= b \\ \lambda_{2,3} &= \frac{1}{2}\left((a+c) \pm \sqrt{(a-c)^2 + e^2}\right) \\ &= \frac{1}{2}\left(1 \pm \sqrt{\frac{(\alpha+\gamma)^2}{(\alpha-\gamma)^2} + e^2}\right) \end{aligned}$$

Because of (7) the following holds

$$\frac{(\alpha+\gamma)^2}{(\alpha-\gamma)^2} + e^2 < \frac{(\alpha+\gamma)^2}{(\alpha-\gamma)^2} - 4\frac{\alpha\gamma}{(\alpha-\gamma)^2} = \frac{(\alpha-\gamma)^2}{(\alpha-\gamma)^2} = 1$$

Hence the eigenvalue b to the eigenvector e_2 is negative and both other eigenvalues are positive. Thus e_2 is the center of q. For completeness we have to show that e_2 really is the saddle of p, i.e. $\alpha \neq \beta$. But this immediately follows if we change the roles of p and q.

First we show that $b < 0$.
To do this we use the ReslTant of P and Q with respect to y (see [WR], p. 24) to show that the common points of p and q would be complex otherwise. We obtain

$$\operatorname{Res}(x,z) = \begin{vmatrix} \alpha x^2 + \gamma z^2 & 0 & \beta & 0 \\ 0 & \alpha x^2 + \gamma z^2 & 0 & \beta \\ a x^2 + exz + c & 0 & b & 0 \\ 0 & ax^2 + exz + c & 0 & b \end{vmatrix}$$

$$= \left((a\beta - \alpha b)x^2 + e\beta xz + (\beta c - b\gamma)z^2\right)^2$$

The term $(a\beta - \alpha b)$ must not vanish. Otherwise two of the common points would be on the line $z = 0$, a contradiction. We can assume $z = 1$. Then the x-components of the common points of p and q are the solutions of the quadratic equation

$$(a\beta - \alpha b)x^2 + e\beta x + (\beta c - b\gamma) = 0$$

which are

$$x_\pm = \frac{-e\beta \pm \sqrt{e^2\beta^2 - 4(a\beta - \alpha b)(\beta c - b\gamma)}}{2(a\beta - \alpha b)}$$

These must be in the real interval $\left[-\sqrt{\frac{-\gamma}{\alpha}}, \sqrt{\frac{-\gamma}{\alpha}}\right]$, otherwise the y-component of the common points would not be real, which can be seen from P. We show that this is only possible if $b < 0$.

First we need the following

Lemma 3:
$2\beta - \alpha - \gamma < 0$ if and only if $a\beta - \alpha b > 0$.

Proof:
By assumption (1) $\alpha \geq \beta$, therefore $2(\beta - \alpha) = 2\beta - \alpha - \gamma - (\alpha - \gamma) \leq 0$ and hence

$$2\beta - \alpha - \gamma \leq \alpha - \gamma$$

Furthermore we know from (1) $\alpha - \gamma > 0$. Now if $2\beta - \alpha - \gamma < 0$ then

$$\frac{1}{\alpha - \gamma} > \frac{1}{2\beta - \alpha - \gamma}$$

and hence

$$(a\beta - \alpha b) = \alpha\beta\left(\frac{1}{\alpha - \gamma} - \frac{1}{2\beta - \alpha - \gamma}\right) > 0$$

In the same way we yield $(a\beta - \alpha b) \leq 0$ if $2\beta - \alpha - \gamma > 0$. Since $(a\beta - \alpha b) \neq 0$ we have $(a\beta - \alpha b) < 0$.□

Next we calculate the values of e, when the $x_\pm$ reach the border of the permissible region, i.e. where $x^2 = \frac{-\gamma}{\alpha}$

Lemma 4:
If $x_\pm = \sigma\sqrt{\frac{-\gamma}{\alpha}}$, with $\sigma = \pm 1$, then

$$\sigma e = -2\frac{\sqrt{-\alpha\gamma}}{\alpha - \gamma}$$

Proof:

We have to solve the radical equation

$$\frac{-e\beta \pm \sqrt{e^2\beta^2 - 4(a\beta - \alpha b)(\beta c - b\gamma)}}{2(a\beta - \alpha b)} = \sigma\sqrt{\frac{-\gamma}{\alpha}}$$

$$\pm\sqrt{e^2\beta^2 - 4(a\beta - \alpha b)(\beta c - b\gamma)} = 2(a\beta - \alpha b)\,\sigma\sqrt{\frac{-\gamma}{\alpha}} + e\beta$$

$$e^2\beta^2 - 4(a\beta - \alpha b)(\beta c - b\gamma) = 4(a\beta - \alpha b)^2\,\sigma^2\frac{-\gamma}{\alpha} + e^2\beta^2 + 4(a\beta - \alpha b)\,\sigma\sqrt{\frac{-\gamma}{\alpha}}\,e\beta$$

Since $(a\beta - \alpha b) \neq 0$ and $\sigma^2 = 1$ we observing that under regard of $0 < \alpha = \sqrt{\alpha^2}$ and $0 > \gamma = -\sqrt{\gamma^2}$

$$\sigma e = \frac{1}{\beta}\left(-\sqrt{\frac{-\gamma}{\alpha}}(a\beta - \alpha b) - \sqrt{\frac{\alpha}{-\gamma}}(\beta c - b\gamma)\right)$$

Now we insert the values for a, b and c and obtain

$$\begin{aligned}(a\beta - \alpha b) &= \frac{\alpha\beta}{\alpha - \gamma} - \frac{\alpha\beta}{2\beta - \alpha - \gamma}\\ (\beta c - b\gamma) &= \frac{\beta\gamma}{\gamma - \alpha} - \frac{\beta\gamma}{2\beta - \alpha - \gamma}\end{aligned}$$

$$\sigma e = \frac{1}{\beta}\left(-\sqrt{\frac{-\gamma}{\alpha}}\alpha\beta\left(\frac{1}{\alpha - \gamma} - \frac{1}{2\beta - \alpha - \gamma}\right) - \sqrt{\frac{\alpha}{-\gamma}}\beta\gamma\left(\frac{1}{\gamma - \alpha} - \frac{1}{2\beta - \alpha - \gamma}\right)\right)$$

and therefore

$$\sigma e = -2\frac{\sqrt{-\alpha\gamma}}{\alpha - \gamma}$$

as proposed.□

Lemma 5:

$(a\beta - \alpha b) > 0.$

Together with lemma 3 we obtain $2\beta - \alpha - \gamma < 0$ and hence $b < 0$ as proposed.

Proof:

Suppose $(a\beta - \alpha b) < 0$. Then the map

$$e \mapsto x_-(e) - \sqrt{\frac{-\gamma}{\alpha}}$$

with $x_-(e) = \frac{-e\beta - \sqrt{e^2\beta^2 - 4(a\beta-\alpha b)(\beta c - b\gamma)}}{2(a\beta - \alpha b)}$ would become positive for sufficiently large e. By lemma 4 the only zero of this function is $e = -2\frac{\sqrt{-\alpha\gamma}}{\alpha-\gamma}$. Now if $x_-(e) \leq \sqrt{\frac{-\gamma}{\alpha}}$ is satisfied we necessarily must have

$$e \leq -2\frac{\sqrt{-\alpha\gamma}}{\alpha - \gamma} < 0$$

On the other hand the function $e \mapsto x_+(e) + \sqrt{\frac{-\gamma}{\alpha}}$ becomes negative if e is less than an appropriate negative number. If $x_+(e) \geq -\sqrt{\frac{-\gamma}{\alpha}}$ is satisfied we must have

$$e \geq 2\frac{\sqrt{-\alpha\gamma}}{\alpha - \gamma} > 0$$

a contradiction. Since $(a\beta - \alpha b) \neq 0$ the proposition follows.□

In the same way we yield for $(a\beta - \alpha b) > 0$ $e^2 < -4\frac{\alpha\gamma}{(\alpha-\gamma)^2}$, which is statement (7).

Case: $f \neq 0$ und $d = e = 0$

We show: This case is not possible.

Again we calculate the resultant of P and Q with respect to y and set $z = 1$.

$$\mathrm{Res}(x, z=1) = \begin{vmatrix} \alpha x^2 + \gamma z^2 & 0 & \beta & 0 \\ 0 & \alpha x^2 + \gamma z^2 & 0 & \beta \\ ax^2 + c & f & b & 0 \\ 0 & ax^2 + c & f & b \end{vmatrix}$$

$$= \big((a\beta - \alpha b)x^2 + (\beta c - b\gamma)\big)^2 + (\alpha x^2 + \gamma)\beta f^2$$

We obtain a quadratic equation in x^2, which must have two positive solutions (otherwise two or more points common to p and q would not be real). Since the polynomial $\mathrm{Res}(x^2)$ becomes positive for sufficiently large and sufficiently negative x^2, this is only possible if $\mathrm{Res}(0) > 0$. On the other hand $\mathrm{Res}(x^2)$ is strictly monotonically increasing in the interval $[0, \infty)$ and hence would not have any positive root, a contradiction.□

Proof of the first corollary

It can easily be seen from symmetry, that the two conics have either four or none common points. By proposition 1 the two conics cannot intersect each other orthogonally in four points.□

Proof of the second corollary

Suppose q_1 and q_2 also have four points in common. Then we know from proposition 1, that they must not have a common center or a common saddle. Hence (without loss) the center of p is the saddle of q_1 and the saddle of p is the center of q_2. But then the center of q_1 must be the saddle of q_2 by proposition 1. Since the center and the saddle of a conic must be orthogonal (cf. chapter about conics) the stationary points of the three conics coincide. Hence by the second part of proposition 1 the saddle of p must be the center of q_1 and of q_2. Hence q_1 and q_2 have a common center, a contradiction.□

Proof of proposition 2

We already know from corollary 1, that q_1 and q_2 have at most two common points. Thus we just have to deal with the case were these conics have exactly two common points. We show, that then q_1 and q_2 cannot intersect orthogonally.

Because of corollary 1 the two conics cannot have a common center. Thus because of proposition 1 the center of p must be the saddle of q_1 and the saddle of p must be the center of q_2, not of q_1 — but it must be inside of q_1 (see figure 1).

Now let us take a look to these conics. Since each of them intersects p orthogonally in four points, we have from (4) (proof of proposition 1)

$$Q_1 = \frac{\alpha}{\alpha - \beta}x^2 + \frac{\beta}{\beta - \alpha}y^2 + \frac{\gamma}{2\gamma - \alpha - \beta}z^2 + dxy$$

where $d \neq 0$ (otherwise its center would be the center of q_2). Because of symmetry, we can assume $d > 0$.

For Q_2 we either have

$$Q_2 = ax^2 + by^2 + cz^2 \qquad \text{case (A)}$$

for proper coefficients $a > c > 0$, $b < 0$ (see proof of proposition 1 for the case $d = e = f = 0$),
or we have for $e^2 < -4\frac{\alpha\gamma}{(\alpha-\gamma)^2}$ from (5)

$$Q_2 = \frac{\alpha}{\alpha-\gamma}x^2 + \frac{\beta}{2\beta-\alpha-\gamma}y^2 + \frac{\gamma}{\gamma-\alpha}z^2 + exz \qquad \text{case (B)}$$

where $\frac{\beta}{2\beta-\alpha-\gamma} < 0$ (see proof of proposition 1).

Case (A):

For abbreviation we rewrite

$$Q_1 = Ax^2 + By^2 + Cz^2 + Dxy$$

where $A > C > 0$, $B < 0$ and $D \neq 0$ (see proof of proposition 1). Then we define the polynomial of degree two

$$R = \tfrac{1}{2}\nabla Q_1 \cdot \nabla Q_2 = 2aAx^2 + 2bBy^2 + 2cCz^2 + (a+b)Dxy$$

Since q_1, q_2 and r must have two common points by assumption, and since each of such a conic is determined uniquely by three points,

$$R = \nu Q_1 + \mu Q_2$$

(cf. proof of proposition 1). By comparison of coefficients we yield

$$\begin{aligned} \nu a + \mu A &= 2aA \\ \nu b + \mu B &= 2bB \\ \mu D &= (a+b)D \end{aligned}$$

Since $D \neq 0$, $\mu = a + b$. Thus we have from equations 1 and 2

$$\nu = (a-b)A/a = (b-a)B/b$$

and

$$(a-b)(bA + aB) = 0$$

Since $a \neq b$,

$$bA + aB = 0$$

But since $bA < 0$ and $aB < 0$, this is a contradiction. Hence the result follows for case (A).

Case (B):

We look at the hemisphere $y > 0$, which contains one of the two components of each conic. It is divided by the line $x = 0$ into two sectors S_+ $(x > 0)$ and S_- $(x < 0)$. Since $d < 0$, the center of q_1 lies in S_-.
The intersection points of q_1 and q_2 with the line $x = 0$ are given by $z_1^2 = \frac{\beta(2\gamma-\alpha-\beta)}{(\beta-\alpha)\gamma} y_1^2$ and $z_2^2 = \frac{\beta(\gamma-\alpha)}{(2\beta-\alpha-\gamma)} y_2^2$, respectively, for arbitrary d and e. Using the fact, that $\gamma < 0$, $\beta - \alpha < 0$ and $2\beta - \alpha - \gamma < 0$, we can show, that $z_1^2 > z_2^2$, for all α, β, γ. Hence the intersection points of q_2 with the line $x = 0$ are inside of q_1 and two of the common points (if they exist) are contained in S_+ (see figure 1).

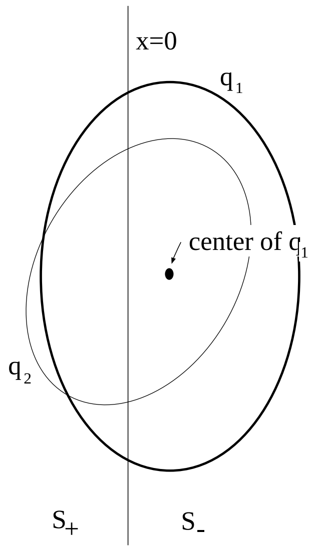


Figure 1

On the other hand q_1 contains the center of q_2 by proposition 1. For this it is necessary, that the sectorial hull of q_1, which is bounded by the tangents of q_1 at these points (= the normals to q_2) contains the center of q_2. Then these common points and the center of q_1 are situated at the same "side" S of the major axis of the ellipse (see figure 2).
This can be seen from the shape of evolute $ev(x)$ of q_2, i.e. the envelope curve of the normals to the conic q_2, which is an astroid (see e.g. [BE2], p. 246–247; to calculate

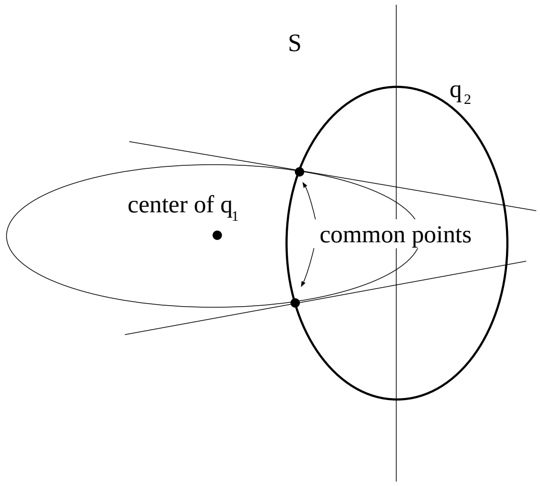


Figure 2

this curve you can use the method described in [BK], p. 99 ff).
Since $e^2 < \infty$, the set S cannot contain the sector S_-, since the common points of q_1 and q_2 are contained in S_+, a contradiction.$\square$

References

[AG] Arfken G.: *Mathematical methods for physicists*, New York–London 1970

[AL] Alessandrini G.: *Critical points of solutions of elliptic equations in two variables*, Ann. della sc. norm. sup. di Pisa **XIV**, 2 (1987), 229–256

[AS] Abramowitz M., Stegun I. A. (eds.): *Handbook of mathematical functions*, Dover–New York 1968

[AR1] Arnol'd V. I.: *The topology of real algebraic curves (the works of Petrovskii and their development)*, (in Russian), Uspekhi Mat. Nauk **28**:5 (1973), 260–262

[AR2] Arnol'd V. I., Oleinik O. A.: *The topology of real algebraic manifolds*, (in Russian), Vestnik Moscov. Univ. **1**:6, 7–17 (1979

[AR3] Arnol'd V. I. et al.: *Some unsolved problems in the theory of differential equations and mathematical physics*, Russ. math. surveys **44**:4, 157–171 (1989)

[BL] Bers L.: *Local behaviour of solutions of general linear elliptic equations*, Commun. Pure Appl. Math. **8**, 473–496 (1955)

[BE1] Berger M.: *Geometry I*, Berlin–Heidelberg 1987

[BE2] Berger M.: *Geometry II*, Berlin–Heidelberg 1987

[BK] Brieskorn E., Knörrer H.: *Ebene algebraische Kurven*, Boston 1981

[BM] Berard P., Meyer D.: *Inegalites isoperimetrique et applications*, Ann. Sci. Ec. Norm. Super., Paris **15** (1982), 513–542

[BO] Boothby W. M.: *An Introduction to Differentiable Manifolds and Riemannian Geometry*, New York–San Francisco–London 1975

[BS] Bronstein I. N., Semendjajew K. A.: *Taschenbuch der Mathematik*, Leipzig 1979

[BW] Blaschke W.: *Projektive Geometrie*, (2. Auflage), Wolfenbüttel–Hannover 1948

[CH] Courant R., Hilbert D.: *Methoden der mathematischen Physik I*, 3. Auflage, Berlin–Heidelberg–New York 1968

[CI] Chavel I: *Eigenvalues in Riemannian Geometry*, New York 1984

[CR] Cigler J., Reichel H.-C.: *Topologie. Eine Grundvorlesung*, Mannheim 1978

[CS] Cheng S. Y.: *Eigenfunctions and nodal sets*, Comment. Math. Helvetici **51**, 43–55 (1976)

[CX] Coexter H. S. M.: *Unvergängliche Geometrie*, Basel 1981

[DA] Durfee A., et al.: *Counting critical points of real polynomials in two variables*, preprint

[DF] Donnelly H., Fefferman C.: *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. math. **93** (1988), 161–183

[DG] Darboux G.: *Principes de géométrie analytique*, Paris 1917

[EE] Enzyklopedie der mathematischen Wissenschaften: *III C.4.15, p. 373*, Leipzig 1921–1928

[FU] Fulton W.: *Intersection Theory*, Berlin–Heidelberg 1984

[GT] Gilbarg D., Trudinger N. S.: *Elliptic partial differential equations of second order*, Berlin–Heidelberg–New York 1977

[GU1] Gudkov D. A.: *The topology of real projective algebraic varieties*, Russian Math. Surveys **29**:4 (1974), 1–79 and Uspekhi Mat. Nauk **29**:4 (1974), 3–79

[GU2] Gudkov D. A.: *On the idea of roughness and degrees of non-roughness for plane algebraic curves*, (in Russian), Mat. Sb. **67**, 4 (1965), 481–527

[GU3] Gudkov D. A.,Utkin G. A., Tai M. L.: *A complete classification of idecomposable curves of fourth order*, (in Russian), Mat. Sb. **69**, 2 (1966), 222–256

[GW] Gordon C., Webb D., Wolpert S.: *Isospectral plane domains and surfaces via Riemannian orbifolds*, Inv. Math. **110**, 1 (1992), 1–22

[HA] Harnack A.: *Über die Vieltheiligkeit der ebenen algebraischen Curven*, Math. Ann. **10** (1876), pp.189–199

[HE] Helgason S.: *Differential Geometry, Lie Groups and Symmetric Spaces*, New York 1978

[HO1] Hoffmann-Ostenhof M.,Hoffmann-Ostenhof T.: *On the local behaviour of nodes of solutions of Schrödinger equations in dimensions* ≥ 3, Comm. PDE **15** (1990), 435–451

[HO2] Hoffmann-Ostenhof M.,Hoffmann-Ostenhof T.: *Local properties of solutions of Schrödinger equations*, Comm. PDE **17** (1992), 491–522

[HS] Hardt R., Simon L.: *Nodal sets for solutions of elliptic equations*, J. of diff. geom. **30** (1989), 505–522

[KA1] Karpushkin V. N.: *Precise order bounds for the number of components of the complement of the zeros of harmonic polynomials*, (in Russian), Funktsional. Anal. i Prilozhen. **19**:4 (1985), 55–60

[KA2] Karpushkin V. N.: *Topology of the zeros of eigenfunctions*, (in Russian), Funktsional. Anal. i Prilozhen. **23**:3 (1989), 59–60

[KF1] Klein F.: *Riemannsche Flächen. Vorlesungen, gehalten in Göttingen 1891/92*, (Nachdruck), Leipzig 1986

[KF2] Klein F.: *Vorlesungen über nicht-euklidische Geometrie*, Berlin 1928

[KR] Kappeler T., Ruf B.: *On the nodal line of the second eigenfunction of elliptic operators in two dimensions*, J. reine angew. Math. **396** (1989), 1–13

[LE] Lewy H.: *On the minimum of domains in which the nodal lines of spherical harmonics divide the sphere*, Commun. PDE **2** (1977), 1233–1244

[LJ] Leydold J.: *Knotenlinien und Knotengebiete, Diploma thesis*, Wien 1989

[MA] Mayeda W.: *Graph Theory*, New York 1972

[ME] Meschkowski H. (ed.): *Meyers Handbuch über die Mathematik*, Mannheim 1967

[MC] Müller C.: *Spherical Harmonics*, Lecture Notes 17, Berlin-Heidelberg-New York 1966

[NN] Nadirashvile N.: *Metric properties of the eigenfunctions of the Laplace operator on manifolds*, Ann. Inst. Fourier, Grenoble **41**,1 (1991), 259–265

[SI] Satake I.: *On the generalization of the notion of manifold*, Proc. Natl. Acad. Sci. USA **42** (1956), 359–363

[SR] Sperb R.: *Maximum Principles and Their Applications*, New York-London 1981

[SW] Stein E., Weiss G.: *Introduction to Fourier analysis on euclidean spaces*, Princeton 1971

[SZ] Szegő G.: *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ. vol. XXIII, rev. ed., New York 1959

[TJ] Thorpe J. A.: *Elementary Topics in Differential Geometry*, New York–Heidelberg–Berlin 1979

[VJ] Vick J. W.: *Homology Theory*, New York–London 1973

[VO] Viro O. Ya.: *Real algebraic plane curves: constructions with controlled topology*, Leningrad Math. J. **1**, 5 (1990), 1059–1134

[VW1] van der Waerden B.L.: *Algebra I*, (8. Auflage, Berlin–Heidelberg–New York 1971

[VW2] van der Waerden B.L.: *Algebra II*, (5. Auflage, Berlin–Heidelberg–New

York 1967

[VW3] van der Waerden B.L.: *Einführung in die algebraische Geometrie*, Berlin 1939

[WH] Whitney H.: *Elementary Structures of Real Algebraic Varieties*, Annals of Math. **66**, 3 (1957), pp.545–556

[WI] Wilson G.: *Hilbert's sixteenth problem*, Topology **17** (1978), 53–73

[WR] Walker R. J.: *Algebraic Curves*, New York–Heidelberg–Berlin 1950

The quotations between the chapters are taken from

Franz Kafka: *Sämtliche Erzählungen*, Frankfurt am Main 1970

Hermann Hesse: *Der Bettler*, Berlin 1951

Peter Rosei: *Reise ohne Ende*, Frankfurt am Main 1983

Some used symbols

θ, φ ..... spherical coordinates
Y_ℓ^m ..... ON-base function
P_ℓ^m ..... associated Legendre polynomials
u .... arbitrary spherical harmonic
U .... harmonic homogeneous polynomials related to u
ℓ .... degree of U and u
p, q ..... algebraic curves
P, Q .... the equations of p and q
A_P ..... symmetric matrix of the quadratic form $\langle x, A_P x\rangle = P(x)$
$\mathcal{N}(u)$ ... nodal set of u
$\mu(u)$ .... number of nodal domains of u
$\varrho_u(x)$ ... the multiplicity of u at x
$\nu_u(x)$ ... $= \varrho_u(x) - 1$
$\nu(u)$ .... $= \sum \nu_u(x)$
$\zeta(u)$ .... number of components of $\mathcal{N}(u)$
$\chi(S^2)$ ... Euler characteristic of the sphere
$\mathcal{G}(u)$ .... graph of u
$\omega(u)$ .... set of vertices of the graph of u
$\mathcal{E}(u)$ .... set of edges of the graph of u
$\Gamma(v, v')$. edge train
$d(\mathcal{G})$ .... diameter of the graph $\mathcal{G}$
$\tilde{E}_\ell$ ...... eigenspace
E_ℓ ...... set of spherical harmonics of degree ℓ with norm 1
$\vec{u}(x)$ .... vector-valued function; its component form an ON-base of the eigenspace
$\vec{v}(x)$,
$\vec{w}(x)$ ... derivatives of $\vec{u}(x)$ w.r.t. spherical coordinates
$\vec{c}_u$ ...... coefficients of u
$d(u)$ .... distance between u and the set of singular spherical harmonics
u_M ..... maximum of u
η .... maximum of normalized spherical harmonics of degree ℓ
$g_\alpha(u)$ ... generalization of distance function $d(u)$

Aber Ihr hättet nicht mehr viel von ihr erfahren, und was Ihr erfahren hättet, hättet Ihr kaum verstanden. Jede Entdeckung wird gleich in die Gesamtheit der Wissenschaften geleitet und hört damit gewissermaßen auf, Entdeckung zu sein, sie geht im Ganzen auf und verschwindet, man muß schon einen wissenschaftlich geschulten Blick haben, um sie dann noch zu erkennen.

Franz Kafka

in: Der Dorfschullehrer

Curriculum vitae

Ich, Josef Leydold, wurde am 14. April 1965 in Krems an der Donau als erstes von zwei Kindern des Tapezierermeisters Josef und der Näherin Stefanie Leydold geboren.
Von 1971 bis 1975 besuchte ich die Volksschule in Hadersdorf am Kamp, danach bis 1983 das Bundesrealgymnasium in Krems an der Donau, Ringstraße 33, naturwissenschaftlicher Zweig, wo ich auch im Mai 1983 maturierte.
Im Herbst 1983 begann ich an der Universität Wien ein Studium der Mathematik und Biochemie. Im Frühjahr 1989 beendete ich mein Mathemetikstudium mit einer Diplomarbeit über "Knotenlinien und Knotengebiete von Eigenfunktionen" unter der Betreuung von Thomas Hoffmann-Ostenhof (Theoretische Chemie, Uni Wien) und Reinhard Bürger (Mathematik, Uni Wien). Danach begann ich an der nun vorliegenden Dissertation zu arbeiten.
Vom 1. Oktober 1990 bis 31. Mai 1991 leistete ich meinen Zivildienst ab (Betreuung geistig- und mehrfachbehinderter Menschen).
Seit 22. September 1990 bin ich mit Elizabeta Leydold-Jenko verheiratet; am 28. April 1992 wurde unser Sohn Miran Aljoša geboren.